TIER ONE ALGEBRA EXAM
August, 1996

The ring of integers is denoted $\mathbb{Z}$. The ring of rational numbers is denoted $\mathbb{Q}$. All rings have identity.

(30)1. Give an example of each of the following (No justification required):
   (a) A nonabelian group of order 18.
   (b) A ring $R$ with exactly four ideals (including $R$ and \{0\}).
   (c) A nonabelian group all of whose proper subgroups are cyclic.
   (d) An element in $S_{11}$ of order 21.
   (e) An infinite group all of whose elements are of finite order.
   (f) A noncommutative ring in which every nonzero element is either a zero divisor or a unit.

(15)2. Let $H, K$ be subgroups of a group $G$ with $K$ normal.
   (a) Prove that $HK = \{hk|h \in H, k \in K\}$ is a subgroup of $G$.
   (b) Provide an example to show that $HK$ need not be a subgroup if neither $H$ nor $K$ is normal.

(15)3. Let $G$ be a group with exactly three subgroups (including $\{e\}$ and $G$).
   (a) Prove that $G$ is a cyclic group.
   (b) Prove that the order of $G$ is $p^2$ for some prime $p$.

(15)4. Let $p$ be a prime and let $R = \{a \in \mathbb{Q}|a = n/m \text{ for } n, m \in \mathbb{Z}, p \nmid m\}$.
   (a) Prove $R$ is a ring under the usual operations in $\mathbb{Q}$.
   (b) Prove that $R$ contains a unique maximal ideal.

(10)5. Let $R$ be a commutative ring. Let $I, J$ be ideals in $R$. Prove that the canonical ring homomorphism $\pi: R \to R/I \oplus R/J$ given by $\pi(r) = (r + I, r + J)$ is an isomorphism if and only if
   (1) $I \cap J = 0$
   and (2) $I + J = R$. 
(10)6. An element $x$ in a ring $R$ is called nilpotent if $x \neq 0$ but $x^k = 0$ for some $k > 0$. Find a necessary and sufficient condition on $n$ for the ring $\mathbb{Z}/n\mathbb{Z}$ to contain a nilpotent element. Prove your answer.

(15)7. (a) Prove that if $F$ is a field then $F$ contains a subfield isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}/p\mathbb{Z}$ for some prime $p$.

(b) Prove that if $F$ is a finite field then there is a prime $q$ such that the number of elements in $F$ is $q^k$ for some positive integer $k$.

(c) Prove there exists a field containing exactly $7^3$ elements.

(10)8. Let $R$ be an integral domain and let $S$ be a subring of $R$. Prove that if $S$ is a field and $R$ is finite dimensional as an $S$–vector space, then $R$ is a field.

(20)9. Let $A$ be the following $3 \times 3$ matrix over $\mathbb{Q}$:

$$
\begin{pmatrix}
0 & 1 & -2 \\
0 & 0 & -2 \\
1 & 0 & 0
\end{pmatrix}
$$

(a) Find the characteristic polynomial of $A$.

(b) Find the minimal polynomial of $A$ over $\mathbb{Q}$.

(c) Let $\mathbb{Q}[A] = \{a_0 I + a_1 A + \ldots + a_k A^k \mid k \geq 0, a_0, a_1, \ldots a_n \in \mathbb{Q}\}$. Prove $\mathbb{Q}[A]$ is a field.
Tier 1 Algebra Examination

August, 1997.

Time: 3 Hours

Question 1 is worth 25 points, questions 2 through 4 are worth 10 points each and questions 5 through 7 are worth 15 points each.

Start each question on a fresh sheet of paper.

1. Give examples (no need to prove anything) or give mathematical reasons if you can not give examples:
   a. An infinite group all of whose elements have orders 1 or 3.
   b. Matrices A and B of sizes 3 x 2 and 2 x 3 respectively such that A \cdot B = I_3,
      (identity matrix of size 3 x 3)
   c. Two elements in the alternating group A_5 which are conjugate in the symmetric group S_5 but not in A_5.
   d. An u.f.d. which is not a p.i.d.
   e. A transcendental element α ∈ ℂ such that α - \frac{1}{α} is an algebraic element.

2. a. Let f be an automorphism of the group ℤ/16ℤ. Show that there exists an odd integer m (1 ≤ m ≤ 15) such that f(x) = mx for all x ∈ ℤ/16ℤ.
   b. Decompose the group Aut(ℤ/16ℤ) of all automorphisms of ℤ/16ℤ as a product of cyclic groups.

3. a. Let G be a group and H be a subgroup. Let N = { g ∈ G | gHg^{-1} = H } be the normalizer of H in G. Show that there is a bijective correspondence between the left cosets of N in G and the set \mathcal{C} = \{ K \mid K = xHx^{-1} for some x ∈ G \} of all conjugates of H.
   b. Let G, H, and N be as in (a) above. If in addition, G is finite with |H| = r and [G:H] = s, then show that the union of all members of \mathcal{C} (i.e. \bigcup_{K ∈ \mathcal{C}} K) has at most (rs - s + 1) elements.
4. Let $V$ be an $n$-dimensional vector space.
   a. Show that a proper subspace $W$ of $V$ is the intersection of all subspaces of $V$ of dimension $n-1$ which contain $W$.
   b. Let $A(V)$ be the vector space of all linear transformations of $V$ to itself. For $x \neq 0$ in $V$, compute the dimension of $A_{\times}(V) = \{ T \in A(V) | T(x) = 0 \}$.

5. Let $R$ be a commutative ring with $1$. For an ideal $I$ of $R$, define $\sqrt{I}$ to be the set $\{ x \in R | x^n \in I \text{ for some integer } n \geq 1 \}$.
   a. Show that $\sqrt{I}$ is an ideal of $R$ which contains $I$.
   b. If $I$ is a prime ideal, show that $\sqrt{I} = I$.
   c. If $R$ is a u.f.d and $x$ is a non-zero, non-unit element in $R$, find $y$ such that $\sqrt{R.x} = R.y$. (Hint: consider a prime power factorization for $x$).

6. Let $R$ be an Euclidean domain with a valuation $v$ (i.e. $v$ is a function from the set of non-zero elements of $R$ to the set of non-negative integers such that (i) $v(x) \leq v(xy)$ for $x, y \in R \setminus \{0\}$ and (ii) given $z \in R$ and $y \in R \setminus \{0\}$, there exist $q$ and $r$ such that $z = yq + r$ with $r = 0$ or $v(r) < v(y)$). Assume further that $v^{-1}(n)$ is finite for all $n$.
   a. Show that for any non-zero ideal $I$, $R/I$ is finite. (Note that $I = R.y$ for some $y$).
   b. For the ring $\mathbb{Z}[i] = \{ a + bi | a, b \in \mathbb{Z} \}$ of Gaussian integers with standard valuation $v$ given by $v(a + bi) = a^2 + b^2$, prove that $\mathbb{Z}[i] / 3 \cdot \mathbb{Z}[i]$ is a field of 9 elements. (Hint: show that 3 is a prime).

7. Let $p$ be a prime and $n$ be a positive integer relatively prime to $p$. Let $K$ be the splitting field of $x^n - 1$ over $\mathbb{F}_p$, the prime field of $p$ elements. Let $[K : \mathbb{F}_p] = m$.
   a. Show that $n$ divides $p^m - 1$. (a hint is given below)
   b. If $r$ is such that $n$ divides $p^r - 1$, show that $m \leq r$.
   (Hint for parts (a) and (b): Show first that roots of $x^n - 1$ are all distinct and they form a subgroup of the multiplicative group $K \setminus \{0\}$ which is cyclic).
   c. Find $[K : \mathbb{F}_3]$ where $K$ is the splitting field of $x^{14} - 1$. 
(24)1. Prove each of the following statements:
(a) If $G$ and $H$ are groups and $f : G \rightarrow H$ is a group homomorphism, then the kernel of $f$ is a normal subgroup of $G$.
(b) If $N$ is a normal subgroup of index 12 in a group $G$ and $g \in G$ with $g^5 \in N$, then $g \in N$.
(c) If $R$ is a commutative ring in which the only ideals are 0 and $R$, then $R$ is a field.
(d) If $I$ and $J$ are ideals in a commutative ring $R$ then the set $S = \{ r \in R | rI \subseteq J \}$ is an ideal of $R$.

(8)2. Determine the number of subgroups of the group $C_5 \times C_5$.

(8)3. Determine the units in the polynomial ring $F_2[x]$, where $F_2$ denotes the ring $Z/2Z$.

(10)4. Determine whether the following matrix is diagonalizable over $R$.

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 5 & 4 \\ -1 & -7 & -5 \end{pmatrix}$$

(10)5. Classify the finite abelian groups with the property that every proper subgroup is cyclic. (A subgroup $H$ of a group $G$ is called proper if $H \neq G$.)

(10)6. Let $R$ be a finite ring, not necessarily commutative. Prove that if $a \in R$ is not invertible then there must exist a nonzero element $b \in R$ such that $ab = 0$.

(10)7. Prove that if $R$ is a commutative ring and the polynomial ring $R[x]$ is a PID, then $R$ must be a field.
(10)8. Let $V$ be a finite dimensional real vector space and let $T : V \to V$ be a linear transformation such that $T^2 = T$.

(a) Prove that $\ker(T)$ and $T(V)$ are complementary subspaces of $V$, that is that $\ker(T) \cap T(V) = 0$ and $\ker(T) + T(V) = V$.

(b) Prove there is a basis of $V$ for which the matrix of $T$ has the following form, where $I_m$ is the $m \times m$ identity matrix and $0_{r,t}$ is the $r \times t$ zero matrix.

\[
\begin{pmatrix}
I_m & 0_{m,s} \\
0_{s,m} & 0_s
\end{pmatrix}
\]

(10)9. Let $F$ be a field and let $K/F$ be a field extension of odd degree. Prove that if $K = F(a)$ for some element $a \in K$, then $F(a) = F(a^2)$. 
1. Give examples (no need to prove anything): (12 points)

   a. Two non-isomorphic abelian groups of order 108 such that the order of every element divides 72.

   b. A linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ such that the eigenvectors of $T$ do not span $\mathbb{C}^3$.

   c. A unique factorization domain $D$ and a pair of elements $u, v \in D$ such that the greatest common
divisor of $u$ and $v$ is NOT a linear combination of $u$ and $v$.

   d. Three ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}_{10}$.

      (A ring homomorphism between rings $A$ and $A'$ is a map $f : A \to A'$ such that for $a, b \in A$, 
      $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.)

2. Let $G$ be a group. An equivalence relation $\equiv$ on $G$ is called a congruence relation if $g_1 \equiv g_2$ and
$h_1 \equiv h_2$ implies $g_1 h_1 \equiv g_2 h_2$. (5 points)

   a. Suppose that $\equiv$ is a congruence relation on $G$. Show that the equivalence class of the identity
   element of $G$ is a normal subgroup of $G$.

   b. Let $\phi : G \to G'$ be a homomorphism of groups and $\equiv'$ be a congruence relation on $G'$. Define
   a relation $\equiv$ on $G$ by: $x \equiv y$ iff $\phi(x) \equiv' \phi(y)$. Show that $\equiv$ is an equivalence relation on $G$ that
   is also a congruence relation. (4 points)

3. Let $A$ be a commutative ring with 1. An element $x \in A$ is called nilpotent if $x^r = 0$ for some
positive integer $r$. (Hint: Factor $u^r - v^r$ and specialize.)

   a. Show that the set $N$ of all nilpotent elements in $A$ is an ideal in $A$ and that the quotient ring
$A/N$ has no non-zero nilpotent elements. (6 points)

   b. Show that if $x \in N$, then $1 - x$ is a unit in $A$. (3 points)
4. Let $P_2(\mathbb{R})$ be the vector space of all polynomials of degree 2 or less with real coefficients. Consider the linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by:

$$T(f)(x) = f(0) + f(1)(x + x^2).$$

Find the eigenvalues of $T$ and determine whether $T$ is diagonalizable. \hspace{1cm} (8 points)

5. Let $G$ be the subgroup of $2 \times 2$ complex invertible matrices generated by $x = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $y = \begin{bmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{bmatrix}$, where $\omega$ is a primitive cube-root of 1. Let $H$ and $K$ be the subgroups of $G$ generated by $x$ and $y$ respectively.

a. Show that $H$ and $K$ are normal in $G$. \hspace{1cm} (6 points)

b. Compute the orders of the subgroups $H, K, H \cap K, \text{ and } G$. \hspace{1cm} (4 points)

6. Let $V$ be an $n$-dimensional real vector space. Consider the set $F(V)$ of all functions from $V \times V$ to $\mathbb{R}$. It can be proved that $F(V)$ is a real vector space under the operations: For all $\phi, \psi \in F(V)$ and $r \in \mathbb{R}$,

(i) $(\phi + \psi)(x, y) = \phi(x, y) + \psi(x, y), \hspace{1cm} (ii) (r\phi)(x, y) = r\phi(x, y)$.

Let $S(V)$ be the subset of all functions $\phi \in F(V)$ which satisfy the following condition: For all $x, y, x' \in V$ and $a, a' \in \mathbb{R}$,

(i) $\phi(x, y) = \phi(y, x), \hspace{1cm} (ii) \phi(ax + a'x', y) = a\phi(x, y) + a'\phi(x', y)$.

a. Show that $S(V)$ is a subspace of $F(V)$ . \hspace{1cm} (3 points)

b. Find the dimension of $S(V)$. (Hint: Use a suitable map from $S(V)$ to the vector space of all $n \times n$ matrices.) \hspace{1cm} (5 points)
7. Consider the ring \( \mathbb{Z}_2[x] \) and two ideals \( I \) and \( J \) generated by the elements \( x^2 - 1 \) and \( x^2 + x + 1 \) respectively.

a. Find all the units in the quotient rings \( \mathbb{Z}_2[x]/I \) and \( \mathbb{Z}_2[x]/J \). \( \text{(7 points)} \)

b. If \( F \) is a field of 4 elements, is it true that \( F \) is isomorphic to one of these two rings? (Justify your answer.) \( \text{(3 points)} \)

8. Find the irreducible polynomial over \( \mathbb{Q} \) of the element \( \alpha = \sqrt{5} \cdot \sqrt[3]{2} \). (Hint: Prove first that \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}, \sqrt[3]{2}) \).) \( \text{(9 points)} \)

9. Let \( R \) be the ring \( \mathbb{Z}[i] \) of Gaussian integers (i.e. \( R = \{a + bi \mid a, b \in \mathbb{Z}\} \).)

a. Let \( p \in \mathbb{Z} \) be a prime integer. Show that \( p \) is a prime element of \( R \) if the equation \( x^2 + y^2 = p \) has no integer solutions for \( x \) and \( y \). (Hint: use the norm \( N(a + bi) = a^2 + b^2 \).) \( \text{(5 points)} \)

b. Using (a) or otherwise, show that 11 does not divide \( 4n^2 + 1 \) for all \( n \in \mathbb{Z} \). \( \text{(5 points)} \)

10. Give reasons why the following examples do not exist: \( \text{(15 points)} \)

a. Elements in \( x, y \in S_5 \) of order 3 and 4 respectively such that \( xy = yx \).

b. Elements \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \) transcendental over \( \mathbb{Q} \) and \( \beta, \alpha^2 + \beta \) both algebraic over \( \mathbb{Q} \).

c. An integral domain with 20 elements.
Each problem is 10 points.

1. Find a invertible matrix $M$ such that $M^{-1}AM$ is diagonal, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

2. Give an example of a $2 \times 2$ matrix that does not have two linearly independent eigenvectors.

3. A square matrix $A$ is called nilpotent if $A^k = 0$ for some $k > 0$.
   a) Give an example with justification of a nonzero nilpotent $A$.
   b) Prove that if $A$ is nilpotent, then $I + A$ is invertible.

4. Let $G$ be a finite group, $a, b \in G$. Prove that the orders of $ab$ and $ba$ are equal.

5. Prove that the set of elements of finite order in an abelian group is a subgroup.

6. Let $G$ be a finite group of order $> 2$. Prove that $G$ has a nontrivial automorphism.

7. Find two generators for the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\{(8, 7), (2, 5), (9, 3)\}$.

8. Consider the ring homomorphism

$$f : \mathbb{Z}[x] \rightarrow \mathbb{R}$$

which maps $x$ to $\sqrt{2}$. Consider the ideal $Ker(f)$ in $\mathbb{Z}[x]$. Show $Ker(f)$ is generated by a single polynomial and find that polynomial.
9. Prove that the ring $H := \mathbb{F}_2[x]/(x^3 + x^2 + 1)$ is a field. Find the degree of the field extension $[H : \mathbb{F}_2]$.

10. Let $R$ be a commutative ring and $I \subset R$ be an ideal. Consider the set

$$J := \{x \in R | x^n \in I \text{ for some } n \geq 1\}.$$

a) Show that $J$ is an ideal in $R$.

An ideal $I$ is called primary if for all $x$ and $y$ satisfying $xy \in I$, either $x \in I$ or $y^m \in I$ for some $m \geq 1$, where $m$ may depend on $y$.

b) Show that if $I$ is primary, $J$ is prime.

11. Show that if some element of a commutative ring has three or more square roots, the ring is not an integral domain.
Tier 1 Examination - Algebra
August 27, 1999

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbb{R}$.

(8)1. Let $G$ be a group and let $H$, $K$ be subgroups of $G$, with $H$ normal. Prove that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of $G$.

(8)2. a. Find the eigenvalues of the following real matrix.

\[
\begin{pmatrix}
0 & 3 & 2 \\
0 & -1 & 0 \\
1 & 3 & 1
\end{pmatrix}
\]

b. Determine whether or not the matrix is diagonalizable over the reals.

(12)3. Let $G$ be a group, $H$ a subgroup of $G$.

a. Prove or disprove: If the index $[G : H] = 2$, then $H$ is normal.

b. Prove or disprove: If the index $[G : H] = 3$, then $H$ is normal.

(8)4. A ring $R$ is called simple if its only (two-sided) ideals are $\{0\}$ and $R$. Prove that if $R$ is simple then the center of $R(= \{x \in R \mid xr = rx \text{ for all } r \in R\})$ is a field.

(8)5. Prove that if $A$ and $B$ are real $n \times n$ matrices, then $AB$ and $BA$ have the same eigenvalues.

(7)6. Let $p$ be a prime number and let $F$ be a field of characteristic $p$. Prove that if $a \in F$ satisfies $a^p = 1$, then $a = 1$.

(8)7. Prove that if $R$ is a principal ideal domain, then every nonzero prime ideal is a maximal ideal. Does the conclusion hold in general for $R$ a unique factorization domain?
(8)8. Let $F$ be a finite field. Prove there is a prime $p$ such that the number of elements in $F$ is $p^k$ for some positive integer $k$.

(15)9. Let $V$ be an $n$-dimensional real vector space. Let $A(V)$ be the set of linear transformations on $V$, that is $A(V) = \{T : V \to V \mid T \text{ is a linear transformation}\}$. Recall that $A(V)$ is also a real vector space, if we define the sum by $(S + T)(v) = S(v) + T(v)$ for $S, T \in A(V)$ and $v \in V$ and the scalar product by $(aT)(v) = aT(v)$ for $a \in \mathbb{R}$ and $T \in A(V)$.

a. What is the dimension of $A(V)$?

b. Let $B(V) = \{T \in A(V) \mid \dim T(V) < n\}$. Determine whether or not $B(V)$ is a subspace of $A(V)$. If it is, find its dimension.

c. Let $W$ be a subspace of $V$ of dimension $k$ and let $C(V) = \{T \in A(V) \mid T(w) = 0 \text{ for all } w \in W\}$. Determine whether or not $C(V)$ is a subspace of $A(V)$. If it is, find its dimension.

(10)10. Let $G = Z_3 \times Z_3$, where $Z_3$ denotes the cyclic group of order 3.

a. Determine the number of distinct homomorphisms from $G$ to itself.

b. Determine the number of distinct isomorphisms from $G$ to itself.

(8)11. Let $R$ be an integral domain and let $S$ be a subring of $R$. Assume $S$ is a field. Then we may view $R$ as an $S$-vector space in a natural way. Prove that if $R$ is finite dimensional as an $S$-vector space, then $R$ is a field.
1a. Let \( G \) be an abelian group of order 60. Consider the homomorphism \( \phi : G \to G \) given by \( \phi(g) = g^5 \). Let \( H = \text{Ker}(\phi) \) and \( K = \text{Image}(\phi) \). Show that \( G \) is the internal direct product of \( H \) and \( K \). (Hint: g.c.d. of integers is a linear combination). 

(8 points)

1b. Would the conclusion in (1a) necessarily be valid for the homomorphism \( \eta \) given by \( \eta(g) = g^{10} \)?

(4 points)

2. A subgroup \( K \) of a group \( G \) is called a characteristic subgroup if for any automorphism \( \theta \) of \( G \), \( \theta(K) = K \).

2a. Show that all subgroups of a cyclic group are characteristic.

(5 points)

2b. Show that the center of a group is a characteristic subgroup.

(5 points)

2c. If \( K \) is a normal subgroup of \( G \) and \( H \) is a characteristic subgroup of \( K \) show that \( H \) is a normal subgroup of \( G \).

(7 points)

2d. Consider the alternating group \( A_4 \). Give example of a characteristic subgroup \( K \) (of \( A_4 \)) and a normal subgroup \( H \) of \( K \) such that \( H \) is not a normal subgroup of \( A_4 \).

(5 points)

3. Let \( K \) be an extension of a field \( F \). If \( \alpha \in K \) is transcendental over \( F \), show that so is \( \beta = \alpha^2 + \frac{1}{\alpha^2} \).

(5 points)

3b. Let \( K \) be an extension of \( F \) of degree \( n \). Let \( f \) be an irreducible polynomial in \( F[x] \) of degree \( m \). If the g.c.d. of \( m \) and \( n \) is 1, show that \( f \) remains irreducible when considered as a polynomial in \( K[x] \). (Hint: consider a root \( \alpha \) of \( f \) in an algebraic closure \( \overline{F} \) of \( F \) which contains \( K \).)

(7 points)

4. Is it possible to have a finite field which is algebraically closed? Justify your answer.

(4 points)

4b. Let \( E \) be an extension of \( \mathbb{Z}_p \) contained in an algebraic closure \( \overline{\mathbb{Z}_p} \). Let \( f \) be an irreducible polynomial in \( \mathbb{Z}_p[x] \) and let \( \alpha, \beta \in \overline{\mathbb{Z}_p} \) be roots of \( f \). If \( \alpha \in E \), show that \( \beta \in E \).

(6 points)
5. For a ring \( R \) with unity \( 1 \), an element \( r \in R \) is said to be a unit if there exists an element \( s \in R \) such that \( rs = 1 = sr \).

5a. Find all the units of the ring \( \mathbb{Z}[i] = \{m + ni \in \mathbb{C} \mid m, n \in \mathbb{Z}\} \). (Hint: think of modulus of a complex number) (5 points)

5b. Give example of a ring with exactly 20 units. (5 points)

5c. Let \( \mathbb{C}[[x]] \) be the ring of formal power series, i.e.
\[
\mathbb{C}[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{C}\} \text{ with addition and multiplication given by }
\]
\[
(\sum_{i=0}^{\infty} a_i x^i) + (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \text{ and }
\]
\[
(\sum_{i=0}^{\infty} a_i x^i) \cdot (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} c_i x^i, \text{ where } c_i = \sum_{j+k=i} a_j b_k.
\]

Show that an element \( r = \sum_{i=0}^{\infty} a_i x^i \) is a unit in \( \mathbb{C}[[x]] \) if and only if \( a_0 \neq 0 \). (5 points)

5d. Let \( I \) be a non-zero ideal in \( \mathbb{C}[[x]] \). Show that there exists a positive integer \( k \) such that \( x^k \in I \). Show further that \( I \) is a principal ideal. (7 points)

6a. Let \( A \) be an \( n \times m \) matrix and let \( B \) be \( m \times n \) matrix with real coefficients such that \( A \cdot B = I_n \), the identity matrix of size \( n \). What is the relationship between \( n \) and \( m \)? (Justify your answer). Further, if \( n = m \), show that \( B \cdot A = I_n \) as well. (Hint: think of corresponding linear transformations of \( \mathbb{R}^n \)). (7 points)

6b. Let \( T \) be a linear transformation of a finite dimensional vector space over \( \mathbb{R} \). Let \( V_1 \) (respectively \( V_{-1} \)) denote the eigenspace of \( T \) for the eigenvalue 1 (respectively -1). If \( T^2 = I_d \), show that \( V \) is a direct sum of \( V_1 \) and \( V_{-1} \). (Hint: think of \( v + T(v) \)). (7 points)

6c. Give an example of a \( 4 \times 4 \) matrix with real entries whose real eigenvalues are \( \pm 1 \) and whose complex eigenvalues are \( \pm 1 \) and \( \pm i \) (no need to justify). (3 points)

6d. Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \) and \( W, W' \) be two subspaces of \( V \). Show that \( \dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W') \). (5 points)
Tier One Algebra Exam
August, 2000

1. (14 points) Let \( A \) be the \( 3 \times 3 \) matrix all of whose entries are 1, i.e.

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

(i) Find the characteristic polynomial of \( A \).
(ii) Find the minimal polynomial of \( A \).
(iii) Find all eigenvalues of \( A \).
(iv) Is \( A \) diagonalizable? If the answer is yes, find \( P \) such that \( PAP^{-1} \) is diagonal. If the answer if no, provide a reason.

2. (8 points) Let \( T \) and \( S \) be linear transformations from \( \mathbb{R}^m \) to \( \mathbb{R}^n \). The coincidence set for \( T \) and \( S \) is defined to be the set \( C(T, S) = \{ w \in \mathbb{R}^m \mid T(x) = w = S(y), \text{ for some } x, y \in \mathbb{R}^m \} \). Let \( T \) and \( S \) be the linear transformations from \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \) represented by the \( 3 \times 4 \) matrices

\[
T = \begin{pmatrix}
1 & 2 & 0 & -1 \\
5 & 4 & 1 & 0 \\
3 & 0 & 1 & 2
\end{pmatrix}
\quad \text{and} \quad
S = \begin{pmatrix}
1 & 2 & 1 & 0 \\
-1 & 0 & 3 & 2 \\
-1 & 4 & 11 & 6
\end{pmatrix}
\]

Find a basis for \( C(T, S) \).

3. (12 points) Let \( G \) be a group.
   (i) Prove that the intersection of a family \( F = \{ H_t : t \in T \} \) of subgroups of \( G \) is a subgroup.
   (ii) Let \( H \) be a subgroup of \( G \). Prove that the intersection 
   \( K = \cap \{ gHg^{-1} : g \in G \} \) of all conjugates of \( H \) is a normal subgroup.
   (iii) Let \( H \) and \( K \) be as in (ii). If \( [G : H] < \infty \), prove that \( [G : K] < \infty \).
4. (6 points) Give an example of a nonabelian group with every proper subgroup abelian.

5. (6 points) Let $S_{10}$ denote the symmetric group in 10 letters. Find the smallest $n$ such that $a^n = 1$, for all $a \in S_{10}$. Justify your answer.

6. (10 points)
   (i) Let $G$ be an abelian group of order $mn$, where $m$ and $n$ are relatively prime. Let $H := \{g \in G : |g| \text{ divides } m\}$ and $K := \{g \in G : |g| \text{ divides } n\}$. Prove that the homomorphism $H \times K \to G$ induced by inclusion is an isomorphism.
   (ii) Let $G$ be an abelian group of order $p_1^{e_1} \cdots p_r^{e_r}$, where $p_i$’s are distinct primes. For each prime $p$ dividing $|G|$, let $G_p := \{g \in G : |g| = p^n \text{ for some } n\}$. Prove that $G \cong G_{p_1} \times \cdots \times G_{p_r}$.

7. (12 points)
   (i) Let $p$ and $q$ be distinct primes. Let $n = pq$. Prove that $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$ as rings.
   (ii) Let $n, p, q$ be as in (ii). Let $\ell$ be any integer. Prove that $a^{1+\ell(p-1)(q-1)} \equiv a \pmod{n}$ for all $a \in \mathbb{Z}$.

8. (4 points) Give three examples of nonisomorphic rings of order 4.

9. (10 points)
   (i) Prove: If $p$ is prime, $f(x) \in \mathbb{Z}_p[x]$ is a polynomial, and $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.
   (ii) Does this remain true if $p$ is not prime? Explain.

10. (10 points)
    (i) Prove that $x^2 - 3$ and $x^5 - 2$ are irreducible in $\mathbb{Q}[x]$.
    (ii) Prove that $x^5 - 2$ is irreducible in $\mathbb{Q}(\sqrt{3})[x]$.

11. (8 points) Write a one or two paragraph essay explaining (without proofs) why trisecting the angle $\pi/3$ is impossible using a straightedge and a compass.
Tier 1 Examination - Algebra

January 4, 2001

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \( \mathbb{R} \). The set of integers modulo \( n \) is denoted \( \mathbb{Z}_n \). The group of permutations on \( n \) letters is denoted \( S_n \).

(20)1. Give an example of each of the following. No justification is required.
(a) A nonabelian group of order 18.
(b) An infinite commutative ring \( R \) such that for all \( y \in R \), \( y + y + y = 0 \).
(c) A 3 by 3 real matrix that is diagonalizable over the complex numbers but not over the reals.
(d) A unique factorization domain that is not a principal ideal domain.
(e) An element of order 3 in \( \text{GL}_2(\mathbb{R}) \).

(10) 2. Let \( G \) be a group with the property that \( g^2 = e \) for all \( g \in G \). Prove \( G \) is abelian.

(10) 3. Determine the number of homomorphisms from \( S_3 \) to \( \mathbb{Z}_2 \times \mathbb{Z}_4 \).

(10) 4. Find \( \lim_{n \to \infty} \left( \begin{array}{cc} 2 & 3 \\ -1/2 & -1/2 \end{array} \right)^n \).

(10) 5. Let \( n \) be a positive odd integer and let \( A, B \in M_n(\mathbb{R}) \) such that \( A^2 = B^2 = I \). Prove that \( A \) and \( B \) have a common eigenvector (not necessarily with the same eigenvalue).

(10) 6. Let \( R \) and \( S \) be commutative rings and let \( \phi : R \to S \) be a ring homomorphism. Suppose there is an ideal \( I \) of \( R \) such that \( \ker(\phi) \subseteq I \subseteq R \) (proper containments). Prove that the image of \( \phi \) is not a field.

(10) 7. Let \( F \) be a subfield of \( \mathbb{R} \) and suppose \( m \) and \( n \) are positive integers with \( \sqrt{m} + \sqrt{n} \in F \). Prove that \( \sqrt{m} \) and \( \sqrt{n} \) are in \( F \).

(10) 8. Let \( K \) be a field and let \( K^\times \) denote the group of nonzero elements of \( K \). Prove that \( K^\times \) contains at most two elements of order 6.
(10)9. Let $G = (\mathbb{Q}, +)/(\mathbb{Z}, +)$, where $(\mathbb{Q}, +)$ denotes the group of rational numbers under addition and $(\mathbb{Z}, +)$ denotes the subgroup of integers under addition. Prove that $G$ is an infinite group in which every element has finite order.

(15)10. (a) Let $R$ be a commutative ring and suppose $I$ and $J$ are ideals of $R$ such that $I + J = R$, where $I + J = \{i + j | i \in I, j \in J\}$. Prove the map $\phi : R/I \cap J \rightarrow R/I \times R/J$ given by $\phi(r + I \cap J) = (r + I, r + J)$ is an isomorphism of rings.

(b) Let $R$ be a commutative ring containing exactly 4 ideals (including $\{0\}$ and $R$). Let $I$ and $J$ denote the other two ideals and suppose they are incomparable, that is $I \not\subseteq J$ and $J \not\subseteq I$. Prove that $R$ is isomorphic to the direct product of two fields.
Algebra Tier 1 Examination
August 2001
Time: 4 hours

Notation:

\( \mathbb{R} \) : field of real numbers   \( \mathbb{R}^n \) : Euclidean n-space
\( \mathbb{Z}_n \) : ring of integers modulo n
\( \mathbb{Z} \) : ring of integers
\( \mathbb{Q} \) : field of rational numbers

1. Give examples (no need to prove anything) OR give mathematical reasons if you can’t give examples of the following: (20 points)

(a) An integral domain which is not a U.F.D. (unique factorization domain)

(b) A field with 6 elements.

(c) A 5 \times 3 matrix \( P \) and a 3 \times 5 matrix \( Q \) (with entries in \( \mathbb{R} \)) such that \( P \cdot Q = I_5 \) (identity matrix of size 5).

(d) A prime number \( p \) such that \( x^2 - x + 2 \) divides \( 3x^3 - 7x^2 + 40x + 27 \) when considered as elements of \( \mathbb{Z}_p[x] \).

(e) An element of order 15 in \( S_9 \).

2a. Let \( G \) be a finite group of order \( n \). Let \( H \) be a unique subgroup of \( G \) of index \( h \). Show that \( H \) is a normal subgroup. Give example of a finite group \( L \) and at least two of its normal subgroups \( M \) and \( N \) of the same index. (5 points)

2b. Find all elements in \( \mathbb{Z}_7 \) which are squares of other elements. Use this and the natural homomorphism \( f : \mathbb{Z} \rightarrow \mathbb{Z}_7 \) to show that the arithmetic sequence 10, 17, 24, 31, ..., has no perfect squares. (5 points)

2c. Describe all the abelian groups \( G \) of order 500 such that \( g^{50} = e \) (identity of \( G \)) for all \( g \in G \). (Show all the work.) (5 points)

3a. Let \( H \) and \( K \) be cyclic groups of order 6 and 4 respectively with generators \( a \) and \( b \) respectively. Let \( G = H \times K \). Let \( L \) be the subgroup of \( G \) generated by \( (a^5, b^2) \). Find the order of the element \( \alpha = (a, b) \cdot L \) in the factor group \( G/L \). Hence or otherwise conclude that \( G/L \) is cyclic. (8 points)
3b. Let $G$ be a group of order 21 which contains unique subgroups of order 3 and 7. Find the number of elements of order 21 in $G$. (7 points)

4a. Let $R$ be a commutative ring with 1. For any $x \in R$, let $I_x = \{ r \in R \mid rx = 0 \}$. Show that $I_x$ is an ideal in $R$. Find this ideal for $R = \mathbb{Z}_{24}$ and $x = 15$. (5 points)

4b. Let $R$ be a commutative ring with 1 which has exactly 3 ideals $\{0\}, J$ and $R$. If $a \in R$ is not a zero divisor then show that $a$ is a unit. (Hint: If $a$ is not a unit then show first that $R \cdot a = R \cdot a^2 = J$ and then get a contradiction. (8 points)

5a. Show that $x^2 - 4x - 4$ is irreducible in $\mathbb{Z}_5[x]$. (2 points)

5b. Show that $R = \mathbb{Z}_5[x]/I$ (where $I$ is the ideal generated by $x^2 - 4x - 4$) is a field. (You may quote and use any relevant results). Find also the number of elements in $R$. (5 points)

5c. Let $I, R$ be as in $5b$. Let $\alpha = x + I \in R$. For this element, find the additive order (i.e. order as an element of the additive group $(R, +)$) and the multiplicative order (i.e. order as an element of the multiplicative group $(R - \{0\}, \cdot)$). (5 points)

5d. Find elements $a, b \in \mathbb{Z}_5$ such that $(2a + 1)(a + b) = 1$. (5 points)

6a. Let $V$ be a real vector space and let $S = \{ v_1, v_2, \ldots, v_m \}$ be a maximal linearly independent set (i.e. $S$ is not properly contained in another linearly independent set. Show that $S$ is a basis of $V$. (4 points)

6b. Let $V = \mathbb{R}^3$. Let $W = \{ (a, b, c) \in \mathbb{R}^3 \mid a = 2b + 3c \text{ and } b^2 = ac \}$. Determine with justification if $W$ is a subspace of $V$. (4 points)

7. Let $V$ denote the vector space of polynomials of degree at most 1 with real coefficients. Consider the linear transformation $T : V \rightarrow V$ given by $T(f(x)) = -f(1) + f(-3)x$ (where $f(a)$ is the value of the polynomial $f(x)$ at $a$).

(a) Find the matrix of $T$ with respect to the basis $\{1, x\}$ of $V$. (3 points)

(b) Find the eigenvalues of $T$ and a basis for each eigenspace. (5 points)

(c) Is $T$ diagonalizable? Justify your answer. (4 points)
Tier 1 Examination - Algebra
January 3, 2002

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \( \mathbb{R} \) and the set of rational numbers by \( \mathbb{Q} \). The set of integers modulo \( n \) is denoted \( \mathbb{Z}_n \). The order of a set \( S \) is denoted \( |S| \).

(10)1. What is the order of the group of invertible \( 2 \times 2 \) matrices with entries in \( \mathbb{Z}_5 \)?

(10)2. Let \( G \) and \( H \) be finite abelian groups of the same order \( 2^n \). Prove that if for each integer \( m \),
\[
|\{ x \in G \mid x^{2^m} = 1 \}| = |\{ x \in H \mid x^{2^m} = 1 \}|
\]
then \( G \) and \( H \) are isomorphic.

(10)3. Prove or give a counterexample for each of the following:
(a) For every integer \( n \) there is a finite group that cannot be generated by \( n \) elements.
(b) If \( G \) and \( H \) are finite groups such that \( |G| \) and \( |H| \) are relatively prime, then there exists a unique homomorphism from \( G \) to \( H \).
(c) Every quotient group \( H \) of a group \( G \) is isomorphic to a subgroup of \( G \).

(10)4. Prove that there exist algebraic numbers \( \alpha \) and \( \beta \), each of degree 3, such that \([Q(\alpha, \beta) : Q] = 6\).

(10)5. Show that for every prime \( p \) there are exactly \( \frac{p^3 - p}{3} \) irreducible cubic polynomials with leading coefficient 1 over the field \( \mathbb{Z}_p \).

(10)6. Prove or disprove: Every maximal ideal of the real polynomial ring \( \mathbb{R}[x, y] \) is of the form \( (x - a, y - b) \) for some \( a, b \in \mathbb{R} \).

(10)7. Prove that for every positive integer \( n \geq 6 \) which is not prime, there exist integers \( a \) and \( b \) such that the congruence equation \( x^2 + ax + b \equiv 0 \pmod{n} \) has more than two solutions modulo \( n \).
(10)8. Let $V$ be a finite dimensional complex vector space and $T : V \to V$ a linear transformation. Suppose there exists $v \in V$ such that \{v, T(v), T^2(v), \ldots, T^{n-1}(v)\} is a basis for $V$. Show that the eigenspaces of $T$ are all 1-dimensional.

(10)9. Prove that a $5 \times 5$ skew-symmetric matrix $A$ has determinant 0. (Recall that a matrix is called skew-symmetric if the transpose of $A$ is the negative of $A$.)

(10)10. For each of the following conditions on a complex square matrix $M$, determine whether the condition implies that $M$ is diagonalizable.

(a) $M^2 = M$
(b) $M^3 = I$
(c) $M^4 = 0$
(1) Consider the matrix \( A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix} \).

(a) Find an invertible matrix \( P \) such that \( P^{-1}AP \) is a diagonal matrix.
(b) Using the previous part of this problem, find a formula for \( A^n \) where \( A^n \) is the result of multiplying \( A \) by itself \( n \) times.
(c) Consider the sequences of numbers 
\[
\begin{align*}
  a_0 &= 1, \quad b_0 = 0, \quad a_{n+1} = -4a_n + 18b_n, \quad b_{n+1} = -3a_n + 11b_n.
\end{align*}
\]
Use the previous parts of this problem to compute closed formulae for the numbers \( a_n \) and \( b_n \).

(2) Let \( P_2 \) be the vector space of polynomials with real coefficients and having degree less than or equal to 2. Define \( D : P_2 \rightarrow P_2 \) by \( D(f) = f' \), that is, \( D \) is the linear transformation given by taking the derivative of the polynomial \( f \). (You needn’t verify that \( D \) is a linear transformation.)

(3) Give an example of each of the following. (No justification required.)

(a) A group \( G \), a normal subgroup \( H \) of \( G \), and a normal subgroup \( K \) of \( H \) such that \( K \) is not normal in \( G \).
(b) A non-trivial perfect group. (Recall that a group is perfect if it has no non-trivial abelian quotient groups.)
(c) A field which is a three dimensional vector space over the field of rational numbers, \( \mathbb{Q} \).
(d) A group with the property that the subset of elements of finite order is not a subgroup.
(e) A prime ideal of \( \mathbb{Z} \times \mathbb{Z} \) which is not maximal.

(4) Show that any field with four elements is isomorphic to 
\[
\frac{\mathbb{F}_2[t]}{(1 + t + t^2)}.
\]

(5) Let \( \mathbb{F}_p \) denote a field with \( p \) elements, \( p \) prime. Consider the ring
\[
R = \frac{\mathbb{F}_p[x, y]}{< x^2 - 3, y^2 - 5 >}
\]
where \( < x^2 - 3, y^2 - 5 > \) denotes the ideal generated by \( x^2 - 3 \) and \( y^2 - 5 \). Show that \( R \) is not a field.

(6) Let
\[
R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}
\]

\textit{Date:} August 2003.
where \( \mathbb{C}[x, y] \) is the polynomial ring over the complex numbers \( \mathbb{C} \) with indeterminates \( x \) and \( y \). Similarly, let \( S \) be the subring of \( \mathbb{C}[t] \) given by \( \mathbb{C}[t^2, i^3] \).

(a) Prove that \( R \) and \( S \) are isomorphic as rings.
(b) Let \( I \) be the ideal in \( R \) given by the residue classes of \( x \) and \( y \). Prove that \( I \) is a prime ideal of \( R \) but not a principal ideal of \( R \).

(7) Suppose that \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a linear map.

(a) Suppose \( n = 2 \) and \( T^2 = -I \). Prove that \( T \) has no eigenvectors in \( \mathbb{R}^2 \).
(b) Suppose \( n = 2 \) and \( T^2 = I \). Prove that \( \mathbb{R}^2 \) has a basis consisting of eigenvectors of \( T \).
(c) Suppose \( n = 3 \). Prove that \( T \) has an eigenvector in \( \mathbb{R}^3 \). Give an example of an operator \( T \) such that \( T \) has an eigenvector in \( \mathbb{R}^3 \), but \( \mathbb{R}^3 \) does not have a basis consisting of eigenvectors of \( T \).

(8) Let \( p(x) \) and \( q(x) \) be polynomials with rational coefficients such that \( p(x) \) is irreducible over the field of rational numbers \( \mathbb{Q} \). Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) be the complex roots of \( p \), and suppose that \( q(\alpha_1) = \alpha_2 \). Prove that

\[
 q(\alpha_i) \in \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} 
\]

for all \( i \in \{ 2, 3, \ldots, n \} \).

(9) Let \( F \) be a field containing subfields \( F_{16} \) and \( F_{64} \) with 16 and 64 elements respectively. Find (with proof) the order of \( F_{16} \cap F_{64} \).

(10) Let \( G \) be a finite group and suppose \( H \) is a subgroup of \( G \) having index \( n \). Show there is a normal subgroup \( K \) of \( G \) with \( K \subset H \) and such that the order of \( K \) divides \( n! \).

(a) Find a matrix representing the linear function \( D \) in the basis \( \{ 1, x, x^2 \} \).
(b) Determine the eigenvalues and eigenvectors of \( D \).
(c) Determine if \( P_2 \) has a basis such that \( D \) is represented by a diagonal matrix. Why or why not?

(11) Suppose that \( W \) is a non-zero finite dimensional vector space over \( \mathbb{R} \). Let \( T \) be a linear transformation of \( W \) to itself. Prove that there is a subspace \( U \) of \( W \) of dimension 1 or 2 such that \( T(U) \subset U \) (i.e. \( U \) is an invariant subspace. Here \( T(U) \) denotes the set \( \{ T(u) \mid u \in U \} \).)
1. Give an example of each of the following. (No proofs required.)
   (a) A square matrix with real coefficients which is not diagonalizable.
   (b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose minimal polynomial is $x^2 - 2x + 1$.
   (c) An element of the alternating group $A_{10}$ of order 12.
   (d) Four nonisomorphic groups of order 66.
   (e) A commutative ring with a prime ideal which is not maximal.

2. Suppose $p_1, p_2, p_3,$ and $p_4$ are real polynomials of degree 3 or less. Determine whether either of the following two conditions implies that the set $\{p_1, p_2, p_3, p_4\}$ is linearly dependent in the vector space of real polynomials. In each case, offer a proof or a counterexample.
   (a) $p_i(0) = 1$ for all $i$.
   (b) $p_i(1) = 0$ for all $i$.

3. Let $T : U \to V$ be a linear transformation of vector spaces and let $W \subset V$ be a subspace. Prove that $T^{-1}(W)$ is a subspace of $U$ and that
   \[
   \dim(T^{-1}(W)) \geq \dim(U) - \dim(V) + \dim(W).
   \]

4. Construct a field $F$ with 16 elements as a quotient of $\mathbb{F}_2[x]$ and find a polynomial in $\mathbb{F}_2[x]$ whose image $\alpha \in F$ satisfies $\alpha^3 = 1$ and $\alpha \neq 1$. Justify your answer.

5. Let $R$ be an infinite ring and let $\varphi : M_2(\mathbb{Z}) \to R$ be a surjective ring homomorphism. Show $\varphi$ is an isomorphism.

6. Let $H$ and $K$ be subgroups of a group $G$. Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup if and only if $HK = KH$.

7. For each $(x, y) \in \mathbb{Z}^2$, let $\langle(x, y)\rangle$ denote the subgroup of $\mathbb{Z}^2$ generated by $(x, y)$. Express $\mathbb{Z}^2/\langle(x, y)\rangle$ as a direct sum of cyclic groups.

8. Let $\alpha \in \mathbb{C}$ be an algebraic number, i.e. a root of a nonzero polynomial with rational coefficients.
(a) Show that \( \alpha \) is a root of a unique monic irreducible polynomial \( f(x) \) with rational coefficients.

(b) Show \( \mathbb{Q}[\alpha] \) is a field.

(c) Show that the degree of \( f(x) \) equals \([\mathbb{Q}[\alpha] : \mathbb{Q}]\).

(d) Write down \( f(x) \) when \( \alpha = \sqrt{2} + \sqrt{3} \). Justify your answer.

9. Let \( F[x] \) denote the polynomial ring over a field \( F \). Let \( R \) denote the subring \( F[x^2, x^3] \).

(a) Show \( R \) is a field.

(b) Show that the quotient field of \( R \) is \( F(x) \).

(c) Show that \( R \) is not a unique factorization domain.

10. For a \( 2 \times 2 \) matrix \( A \) with real coefficients, define \( e^A \) to be the matrix

\[
\sum_{n=0}^{\infty} \frac{1}{n!} A^n.
\]

Assuming that above series converges and that for all \( 2 \times 2 \) matrices \( B \) and \( C \)

\[
Be^AC = \sum_{n=0}^{\infty} \frac{1}{n!} B A^n C,
\]

compute \( e^A \) where \( A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \).
(1) Give an example of each of the following. (No justification required.)

(a) A group \( G \), a normal subgroup \( H \) of \( G \), and a normal subgroup \( K \) of \( H \) such that \( K \) is not normal in \( G \).
(b) A non-trivial perfect group. (Recall that a group is perfect if it's only it has no non-trivial abelian quotient groups.)
(c) A \( 2 \times 2 \) matrix \( A \) over \( \mathbb{R} \) which is not diagonalizable over \( \mathbb{R} \).
(d) A field which is a three dimensional vector space over the field of rational numbers, \( \mathbb{Q} \).
(e) A group with the property that the subset of elements of finite order is not a subgroup.
(f) A prime ideal of \( \mathbb{Z} \times \mathbb{Z} \) which is not maximal.

(2) An element \( a \) in a ring \( R \) is called idempotent if \( a^2 = a \). Show the the only idempotent elements in an integral domain are 0 and 1.

(3) Consider the matrix \( A = \begin{pmatrix} -4 & 18 \\ -3 & 11 \end{pmatrix} \).

(a) Find an invertible matrix \( P \) such that \( P^{-1}AP \) is a diagonal matrix.
(b) Using the previous part of this problem, find a formula for \( A^n \) where \( A^n \) is the result of multiplying \( A \) by itself \( n \) times.
(c) Consider the sequences of numbers
\[
a_0 = 1, b_0 = 0, a_{n+1} = -4a_n + 18b_n, b_{n+1} = -3a_n + 11b_n.
\]
Use the previous parts of this problem to compute a closed formulae for the numbers \( a_n \) and \( b_n \).

(4) Let
\[
R = \frac{\mathbb{C}[x, y]}{(x^2 + y^3)}
\]
where \( \mathbb{C}[x, y] \) is the polynomial ring over the complex numbers \( \mathbb{C} \) with indeterminates \( x \) and \( y \). Similarly, let \( S \) be the subring of \( \mathbb{C}[t] \) given by \( \mathbb{C}[t^2, t^3] \).

(a) Prove that \( R \) and \( S \) are isomorphic as rings.
(b) Let \( I \) be the ideal in \( R \) given by the residue classes of \( x \) and \( y \). Prove that \( I \) is a prime ideal of \( R \) but not a principle ideal of \( R \).
(5) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.

(a) Suppose $n = 2$ and $T^2 = -I$. Prove that $T$ has no eigenvectors in $\mathbb{R}^2$.
(b) Suppose $n = 2$ and $T^2 = I$. Prove that $\mathbb{R}^2$ has a basis consisting of eigenvectors of $T$.
(c) Suppose $n = 3$. Prove that $T$ has an eigenvector in $\mathbb{R}^3$. (I suggest we omit this last part to shorten the problem: Give an example of an operator $T$ such that $T$ has an eigenvector in $\mathbb{R}^3$, but $\mathbb{R}^3$ does not have a basis consisting of eigenvectors of $T$.)

(6) Suppose that $W$ is a non-zero finite dimensional vector space over $\mathbb{R}$. Let $T$ be a linear transformation of $W$ to itself. Prove that there is a subspace $U$ of $W$ of dimension 1 or 2 such that $T(U) \subset U$ (i.e. $U$ is an invariant subspace. Here $T(U)$ denotes the set $\{T(u) | u \in U\}$.)

(7) Let $p(x)$ and $q(x)$ be polynomials with rational coefficients such that $p(x)$ is irreducible over the field of rational numbers $\mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be the complex roots of $p$, and suppose that $q(\alpha_1) = \alpha_2$. Prove that

$$q(\alpha_i) \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

for all $i \in \{2, 3, \ldots, n\}$.

(8) Let $F$ be a field containing subfields $F_{16}$ and $F_{64}$ with 16 and 64 elements respectively. Find (with proof) the order of $F_{16} \cap F_{64}$.

(9) Let $G$ be a finite group and suppose $H$ is a subgroup of $G$ having index $n$. Show there is a normal subgroup $K$ of $G$ with $K \subset H$ and such that the order of $K$ divides $n!$.

(10) Let $P_2$ be the vector space of degree less than or equal to 2 polynomials with real coefficients. Define $D : P_2 \to P_2$ by $D(f) = f'$, that is, $D$ is the linear transformation given by taking the derivative of the polynomial $f$. (You needn’t verify that $D$ is a linear transformation.)

Find a matrix representing the linear function $D$ in the basis $\{1, x, x^2\}$. Determine the eigenvalues and eigenvectors of $D$. Determine if $P_2$ has a basis such that $D$ is represented by a diagonal matrix. Why or why not?
ALGEBRA TIER I EXAM
January 2004

You have 25 questions. Each one is worth 4 points.

1. Let $V \neq \{0\}$ be a vector space over the real numbers $\mathbb{R}$ with the dimension $\dim(V) = n < \infty$. Let $f : V \to V$ be a linear mapping different from the identical mapping $\text{id}_V$ and different from $-\text{id}_V$. Assume that $f$ is an involution, i.e., $f \circ f = \text{id}_V$.

   a) Prove that the only possible eigenvalues are $\lambda = 1$ and $\lambda = -1$.
   b) Let $V_1$ and $V_{-1}$ be the subspaces of $V$ consisting of all the eigenvectors corresponding to the eigenvalue $\lambda = 1$ and $\lambda = -1$, respectively. Prove that $V_1 \neq \{0\}$, $V_{-1} \neq \{0\}$, $V_{-1} \cap V_1 = \{0\}$, and $V_{-1} + V_1 = V$.
   c) Prove that there exists a basis for $V$ such that the matrix representation $F$ for $f$ takes the form of a $2 \times 2$ block matrix

       $F = \begin{pmatrix}
       I_p & 0_{p \times q} \\
       0_{q \times p} & -I_q
      \end{pmatrix},$

    where $I_p$, $I_q$, $0_{p \times q}$, and $0_{q \times p}$ denote the $p \times p$ unit matrix, the $q \times q$ unit matrix, the $p \times q$ zero matrix, and the $q \times p$ zero matrix, respectively, and where $p + q = n$, with $p > 0$ and $q > 0$.

2. Let $V \neq \{0\}$ be a vector space over the real numbers $\mathbb{R}$ with the dimension $\dim(V) = n < \infty$. Let $f : V \to V$ be a linear mapping. Assume that $f$ has the property $f \circ f = -\text{id}_V$, where $\text{id}_V$ denotes the identity mapping of $V$.

   a) Prove that $f$ has no (real) eigenvalues.
   b) Let $e \in V$, $e \neq 0$. Prove that $e$ and $f(e)$ are linearly independent.
   c) Prove that the dimension $n$ of $V$ must be even.

3. Consider the following matrices in $M_3(\mathbb{C})$.

   \[
   (i) \begin{pmatrix}
   0 & 0 & 1 \\
   1 & 0 & -1 \\
   0 & 1 & 1
   \end{pmatrix},
   (ii) \begin{pmatrix}
   3 & 0 & 0 \\
   2 & 3 & -1 \\
   0 & 0 & 3
   \end{pmatrix},
   (iii) \begin{pmatrix}
   2 & -3 & 1 \\
   1 & -2 & 1 \\
   1 & -3 & 2
   \end{pmatrix}
   \]

   a) Find the characteristic polynomials $C(x)$ for each of the matrices above.
   b) Find the minimal polynomials $M(x)$ for each of the matrices above.
   c) Find the Jordan canonical form for each of the matrices above.

4. a) Describe the automorphism group $\text{Aut}(\mathbb{Z}/9\mathbb{Z})$ of the group $\mathbb{Z}/9\mathbb{Z}$.
   b) Give an example of a group with the trivial center.
   c) Prove that if a group $G$ has the trivial center then $|\text{Aut}(G)| \geq |G|$.

5. Let $S_n$ denote the symmetric group on $n$ letters.

   a) Find an element $a \in S_n$ of order $n$.
   b) Find an element $b \in S_5$ of order 6.
   c) Describe the conjugacy classes in $S_4$. How many are there?
6. Let $G$ be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by elements $(2, 6), (8, 21), (-4, -9)$. Find two generators for $G$.

7. a) Let $R$ be a commutative ring, $I \subseteq R$ an ideal such that $R/I$ is a field. Let $a, b \in R$ such that $ab \in I$. Prove that $a \in I$ or $b \in I$.

b) Describe all maximal ideals in the polynomial ring $\mathbb{C}[x]$.

c) Show that the principal ideal $(y) \subseteq \mathbb{C}[x, y]$ is prime, but not maximal.

d) Show that the principal ideal $(y - x^2) \subseteq \mathbb{C}[x, y]$ is prime.

8. Consider the ring $M_n(\mathbb{R})$ of $n \times n$ matrices with real entries.

a) Show that $M_n(\mathbb{R})$ is a simple ring (has no nontrivial two-sided ideals).

b) Let $M$ be a left module over $M_n(\mathbb{R})$. Notice that the field $\mathbb{R}$ is embedded in the ring $M_n(\mathbb{R})$ as scalar matrices; hence in particular $M$ is an $\mathbb{R}$-vector space. Prove that $\dim_\mathbb{R}(M) \geq n$. (Hint: use a)).

9. a) Let $\mathbb{F}_q$ denote a finite field with $q$ elements. Prove that $q = p^n$ for a prime $p$ and some $n > 0$. (You cannot use the classification of finite fields.)

b) Describe the Galois group $Gal(\mathbb{F}_8/\mathbb{F}_2)$ explicitly.

c) Let $f(x) \in \mathbb{F}_2[x]$ be a polynomial of degree 5. Show that the field $\mathbb{F}_{32}$ contains all the roots of $f(x)$. 
1. Let $A$ be a $5 \times 5$ matrix with entries in $\mathbb{R}$. If $A^3 = 0$ and 
\[ \dim(\ker(A)) + \dim(\ker(A^2)) = 8, \]
find the Jordan canonical form of $A$.

2. Let $e_1, e_2$ be two (non-zero) eigenvectors of a linear transformation $T : V \rightarrow V$, and assume $e_1 \neq -e_2$. Show that $e_1 + e_2$ is an eigenvector of $T$ if and only if $e_1, e_2$ correspond to the same eigenvalue.

3. Given that the characteristic polynomial of a $5 \times 5$ integer matrix is $x(x^4 + 1)$, is the matrix diagonalizable over $\mathbb{R}$? over $\mathbb{C}$? In each case, give its diagonal form if it exists or explain why it is not diagonalizable.

4. List all the abelian subgroups of $S_4$, the permutation group on 4 elements.

5. Is the group of automorphisms of an abelian group always abelian? Prove, or give a counterexample.

6. Let $\mathbb{C}^*$ denote the group whose elements are $\mathbb{C} \setminus \{0\}$, equipped with multiplication.

(a) Show that for any group $G$ and any abelian group $H$, the group operation of $H$ induces an operation on $\text{Hom}(G, H)$ which makes it into an abelian group.

(b) Use the structure theorem for finite abelian groups to show that for any finite abelian group $G$, $\text{Hom}(G, \mathbb{C}^*)$ is isomorphic to $G$.

7. Let $H, K$ be proper subgroups of a group $G$, so that neither of them is contained in the other one. In each of the following two cases, prove that $H \cap K$ must be normal in $G$ or prove that it cannot be normal in $G$ or give examples showing that it can either be normal or not normal. (i) $H, K$ both normal in $G$. (ii) $H$ is normal in $G$, but $K$ is not.

8. Let $R$ be an integral domain. Prove:

(a) An element $a \in R$ is irreducible (meaning that $a$ is not a unit, and whenever $a = bc$ for $b, c \in R$, either $b$ or $c$ is a unit) if and only if $Ra$ is maximal among the proper principal ideals of $R$.

(b) If $R$ is a principal ideal domain, then each irreducible element in $R$ is prime.

9. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

10. Let $R = F[x]$, where $F$ is a field. Let $f(x)$ and $g(x)$ be polynomials in $R$ such that the degree of $f(x)$ is smaller than the degree of $g(x)$. Let $g(x) = g_1(x) \cdot g_2(x)$ for relatively prime polynomials $g_i(x)$. 
(7 pts) (a) Show that there are polynomials \( f_i(x) \) such that the degree of each \( f_i(x) \) is smaller than the degree of \( g_i(x) \) for \( i = 1, 2 \) and such that
\[
\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)}.
\]

(4 pts) (b) Are the \( f_i(x) \) unique? Justify your answer.

(4 pts) 11. Let \( p \) be a prime and let \( \alpha \) be algebraic over \( \mathbb{Z}_p \), the field with \( p \) elements. It is given that the multiplicative order of \( \alpha \) is \( k \).
Show that \( p \) does not divide \( k \).

(6 pts) 12. Find the irreducible polynomial of \( \alpha = \sqrt{2} + \sqrt[3]{5} \) over \( \mathbb{Q} \). Justify why the polynomial you give is \( \alpha \)'s irreducible polynomial.

13. Let \( p \) be a prime. For a natural number \( r \), let \( GF(p^r) \) denote the finite field with \( p^r \) elements.

(4 pts) (a) Prove one of the two implications in the following proposition:

Proposition. \( GF(p^r) \subseteq GF(p^s) \) if and only if \( r \) divides \( s \).

(7 pts) (b) Assuming the proposition above, prove that there are \( p^{15} - p^3 - p^5 + p \) elements \( \alpha \) in the algebraic closure of \( \mathbb{Z}_p \) for which \( \mathbb{Z}_p(\alpha) \) is \( GF(p^{15}) \).

14. Let \( K \) be an extension field of the field \( F \), and let \( \alpha \) be an element of \( K \).

(5 pts) (a) If \( \alpha \) is transcendental, show that \( F(\alpha) \neq F(\alpha^2) \). Give an example to show that the reverse implication is not true.

(5 pts) (b) If \( \alpha \) is algebraic over \( F \) and \( F(\alpha) \neq F(\alpha^2) \), show that \([F(\alpha) : F] \) is even. Give an example to show that the reverse implication is not true.
(1) (a) Compute the units in the ring $\mathbb{Z}_4[x]$.

(b) Find an irreducible polynomial of degree 4 in $\mathbb{Z}_2[x]$. Justify your answer.

(2) Give examples of two $6 \times 6$ matrices $A$ and $B$ over $\mathbb{Q}$ with minimal polynomial $(x - 2)^2(x^2 + 3)$ such that $A$ is not similar to $B$.

(3) Find an explicit formula for the entries of the following matrix in terms of $n$.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

(4) Let $G$ be a group containing a normal subgroup $H$ isomorphic to $D_8$, the dihedral group of order 8. Prove that $G$ must have a nontrivial center.

(5) Let $\phi$ be a group homomorphism from $D_{18}$, the dihedral group of order 18, to $S_8$. Prove that $\phi$ is not injective.

(6) Let $R$ be a unique factorization domain.

(a) Prove that if $P$ is a nonzero prime ideal of $R$ then $P$ must contain an irreducible element.

(b) A prime ideal $P$ in an integral domain $R$ is called minimal if $P \neq 0$ and the only prime ideals $Q$ such that $Q \subseteq P$ are $Q = 0$ and $Q = P$. Prove that in a unique factorization domain every minimal prime is principal.

(7) Let $F$ be a field and let $f(x) \in F[x]$ be a nonzero polynomial. Let $n$ be the degree of $f(x)$. The quotient ring $F[x]/(f(x))$ may be viewed as an $F$–vector space via $\alpha(g(x) + (f(x))) = \alpha g(x) + (f(x))$ for all $\alpha \in F$ and $g(x) \in F[x]$. Prove that this vector space is finite dimensional over $F$ and that the images of the elements $1, x, x^2, \ldots, x^{n-1}$ form a basis.
Let $R$ be a ring and let $N(R) = \{ r \in R | r^k = 0 \text{ for some } k > 0 \}$.

(a) Prove that if $R$ is commutative, then $N(R)$ is an ideal in $R$.

(b) Give an example to show that there are noncommutative rings $R$ in which $N(R)$ is not an ideal.

Prove that for any prime $p$, the field $\mathbb{F}_p$ with $p$ elements contains an element $a$ such that $[\mathbb{F}_p(\sqrt[p]{a}) : \mathbb{F}_p] = 3$ if and only if $p - 1$ is divisible by 3.

For each of the following construct an example or prove that there is none.

(a) A finite field extension $L/K$ and elements $\alpha, \beta \in L$ of degree 2 over $K$ such that $\alpha + \beta$ is of degree 3 over $K$.

(b) A finite field extension $L/K$ and elements $\gamma, \delta \in L$ of degree 3 over $K$ such that $\gamma + \delta$ has degree 6 over $K$. 
The point value for each problem is indicated. For problems with multiple parts, each part carries an equal point value unless indicated otherwise. Please provide an explanation for your answer to each problem, unless indicated otherwise.

1. (8 points) Given finite groups $G_1, G_2$ and $H$, along with surjective homomorphisms $\phi_i : G_i \to H$, define
   \[ P = \{(g_1, g_2) \in G_1 \times G_2 \mid \phi_1(g_1) = \phi_2(g_2)\}. \]
   a. Show that $P$ is a subgroup and that the homomorphism $f : P \to G_1$ given by $f(g_1, g_2) = g_1$ is surjective.
   b. Find a relation between the orders of $G_1, G_2, H,$ and $P$.

2. (8 points)
   a. What is the largest order of a cyclic subgroup of the symmetric group $S_7$?
   b. How many cyclic subgroups are there of the order found in part (a)?

3. (10 points) Let $\mathbb{Z}_n$ denote the additive cyclic group of order $n$. Let $G$ be the group $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{11}$ and let $H$ be the subgroup $\{6g \mid g \in G\}$.
   a. Express the groups $H$ and $G/H$ as direct sums of cyclic groups.
   b. Is $G$ isomorphic to $H \oplus G/H$?

4. (10 points) Let $G$ be a group and $H$ a subgroup of $G$. Suppose further that $g^2 \in H$ for all $g \in G$.
   a. Show that $H$ is a normal subgroup of $G$.
   b. Give an example of a group $G$ with a non-normal subgroup $K$ such that $g^4 \in K$ for all $g \in G$.

5. (6 points) Find polynomials $u(x)$ and $v(x)$ in $\mathbb{Q}[x]$ such that
   \[ u(x)(x^3 - 5) + v(x)(x^2 + 2x + 3) = 1. \]

6. (6 points) Let $R$ be any ring with 1 such that $r^2 = r$ for all $r \in R$. Show that the characteristic of $R$ is 2 and that $R$ is commutative.
7. Let $D$ be an integral domain that is not a field.
   a. (2 points) Show that $D$ has a nonzero proper ideal.
   b. (6 points) Show further that $D$ has infinitely many distinct ideals.

8. (6 points) Consider two subspaces $W_1, W_2$ of $\mathbb{R}^3$, each spanned by 2 vectors: $W_1$ by \{$(1,2,1), (-1,-1,0)$\} and $W_2$ by \{$(2,3,3), (1,1,-2)$\}. Find a basis for $W_1 \cap W_2$.

9. (6 points) Show that there does not exist a $4 \times 4$ matrix $A$ with real entries which satisfies all of the following properties: (i) the characteristic polynomial is $(x + 2)^4$, (ii) $(A + 2I)^2 \neq 0$, and (iii) rank $(A + 2I)^2 = \text{rank} (A + 2I)^3$.

10. (8 points) Let $V$ be a vector space of dimension greater than 1 over the real numbers. Let $T$ be a linear transformation of $V$ to itself of rank 1. Show that the minimal polynomial of $T$ is $x(x - a)$ for some real number $a$.

11. (6 points) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^5 + 1$. If $\alpha \neq -1$, show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

12. a. (6 points) Construct a field of 27 elements and find the structure of its additive group.

   b. (4 points) State without proof the structure of the multiplicative group of nonzero elements in that field.

13. (8 points) For each of the following, construct an example. No explanation is required.
   (a) An infinite field of characteristic $p \neq 0$.
   (b) A field $K$ and a proper subfield $E$, both of which are algebraically closed.
ALGEBRA TIER I
January 2006

Unless stated otherwise, all your answers require justification. A correct answer without a correct proof earns little credit. All questions are worth the same number of points.

1. Find the eigenvalues of the complex matrix

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

2. Consider the complex matrices

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

a) Find characteristic polynomials of \( A \) and \( B \).
b) Does there exist an invertible matrix \( P \) such that \( P A P^{-1} = B \)?

3. Consider the complex matrices

\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
D = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

a) Find the ranks of \( C \) and \( D \).
b) Does there exist an invertible matrix \( P \) such that \( P C P^{-1} = D \)?

4. Let \( A \) be a \( n \times n \) complex matrix such that \( A^2 \) is the identity matrix. Prove that there exists an invertible \( n \times n \) matrix \( Q \) such that the matrix \( QAQ^{-1} \) is diagonal.

5. Let \( A \) be an invertible matrix, and let \( E \) be an upper triangular matrix with zeroes on the diagonal, that is: the \((i,j)\)'th entry of \( E \) is 0 for all \( j < i \). Assume that \( AE = EA \). Show that the matrix \( A + E \) is invertible.
6. Give an example or compute, without proof, each of the following:
   a) All the abelian groups, up to isomorphism, of order 144 (list them).
   b) An infinite group all of whose elements have finite order.
   c) Groups $H \triangleleft K \triangleleft G$ ($H$ is normal in $K$ and $K$ is normal in $G$) such that $H$ is not normal in $G$.

7. Let $G$ be a group, and let $H$ and $K$ be normal subgroups with $H \cap K = \{e\}$. If $h \in H$ and $k \in K$ then show that $hk = kh$.

8. Prove that the center of the finite symmetric group $S_n$ is $\{e\}$ for $n \geq 3$. (Recall that the center is the set of elements which commute with every element of the group.)

9. How many conjugacy classes are there in the symmetric group $S_5$?

10. Show that if $G$ has trivial center, then $G$ is isomorphic to a subgroup of the group of automorphisms of $G$. (An automorphism of $G$ is an isomorphism $\phi : G \rightarrow G$. Note that the set of automorphisms forms a group under composition.)

11. What are the units in the ring of Gaussian integers $\mathbb{Z}[i]$?

12. a) Let $S$ be a subset of the complex numbers $\mathbb{C}$. Show that the intersection of all the fields $F \subseteq \mathbb{C}$ which contain $S$ is a field.

   b) Let $F_1, F_2$ be subfields of $\mathbb{C}$, and assume that $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}] = 2$ and $F_1 \neq F_2$. Let $F_3$ be the minimal subfield of $\mathbb{C}$ which contains $F_1 \cup F_2$. Show that $[F_3 : \mathbb{Q}] = 4$.

13. a) Let $F$ be a finite field of order $p^a$. Show that every non-zero element $x \in F$ satisfies $x^{p^a-1} = 1$.

   b) Explain why the polynomial $x^n - 1$ can have at most $n$ roots in $F$.

   c) Show that the non-zero elements of $F$, endowed with $F$'s multiplication, form a cyclic group of order $p^a - 1$. 
Tier 1 Examination - Algebra

August 23, 2006

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by \( \mathbb{R} \) and the set of rational numbers by \( \mathbb{Q} \).

(20) 1. Find an example of each of the following (no proof necessary):
(a) An infinite integral domain in which there are exactly 4 units.
(b) Two nonisomorphic nonabelian groups of order 12.
(c) A unique factorization domain with exactly one irreducible element (up to multiplication by a unit).
(d) An element of order 3 in \( GL_2(\mathbb{Q}) \).

(10) 2. Find the sum of the reciprocals of the eigenvalues of the following matrix:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

(10) 3. Let \( n \) be a positive integer. Let \( V_0, V_1, \ldots, V_{2n} \) be a sequence of finite dimensional vector spaces. For \( i = 0, 1, \ldots, 2n \), let \( T_i : V_i \to V_{i+1} \) be linear transformations, where by convention, \( V_{2n} = V_0 \) and \( T_{2n} = T_0 \). Suppose that for \( i = 0, 1, \ldots, 2n-1 \), we have

\[
ker(T_{i+1}) = im(T_i)
\]

Prove that

\[
dim(V_0) + dim(V_2) + dim(V_4) + \cdots + dim(V_{2n-2}) = dim(V_1) + dim(V_3) + \cdots + dim(V_{2n-1}).
\]

(10) 4. Let \( R \) be a ring with unit (possibly non-commutative). An element \( \alpha \) in \( R \) is called left quasi-invertible if \( 1 - \alpha \) is left invertible, that is, if there exists \( b \in R \) such that \( b(1 - \alpha) = 1 \). A subset of \( R \) is called left quasi-invertible if all of its elements are left quasi-invertible.
(a) Show that if \( \alpha \) is in every maximal left ideal, then \( \alpha \) is left quasi-invertible.
(b) Show that if the left ideal generated by $\alpha$ is left quasi-invertible, then $\alpha$ is contained in every maximal left ideal.

(15)5. Let $R$ be a commutative ring. If $I$ and $J$ are ideals in $R$ we define the product ideal to be $IJ = \{\sum_{k=1}^{n} x_k y_k \mid n \geq 1 \text{ and } x_k \in I, y_k \in J\}$ and we define the sum ideal to be $I + J = \{x + y \mid x \in I, y \in J\}$.

(a) Prove that $IJ$ is an ideal in $R$.

(b) Prove that $IJ \subseteq I \cap J$ and give an example to show that equality does not always hold.

(c) Prove that if $I + J = R$ then $IJ = I \cap J$.

(10)6. (a) Let $R$ be an integral domain containing a subring $F$ such that $F$ is a field and such that $R$ is finite dimensional as a vector space over $F$. Show that $R$ is a field.

(b) Let $T$ be a field extension of the field $F$ and let $K$ and $L$ be intermediate fields such that $K$ and $L$ are both finite dimensional over $F$. Let $KL = \{\sum_{k=1}^{n} x_k y_k \mid n \geq 1 \text{ and } x_k \in K, y_k \in L\}$. Prove $KL$ is a subfield of $T$.

(10)7. Let $H$ and $K$ be subgroups of the group $G$. Prove that $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

(10)8. Let $G$ be a group and $x, y$ elements of order 2. Let $H$ be the subgroup generated by $x$ and $y$. Prove that the subgroup generated by $xy$ is normal in $H$ and has index two in $H$.

(10)9. Let $F$ be a field, let $f(X)$ be a polynomial with coefficients in $F$, and let $R = F[X]/(f(X))$.

(a) Suppose $F$ is the rational numbers and $f(X) = X^2 - 1$. Let $\alpha$ be the image of $a_0 + a_1 X + \cdots + a_n X^n$ in $R$ (for $a_0, \ldots, a_n \in F$). Find concise necessary and sufficient conditions on $a_0, \ldots, a_n$ for $\alpha$ to be a unit.

(b) Let $f(X) = X^3 - 3X^2 - 1$. Show that if $F$ is the real numbers, then $R$ has zero divisors, but if $F$ is the rational numbers, then $R$ does not.
Tier 1 Algebra Examination, January, 2007

Important:

• Justify fully each answer unless otherwise directed.

• Notation: \{1, 2, 3, \ldots\} = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} denote respectively the natural numbers, integers, rationals, reals, and complex numbers

1. (5 each pts.)
   (a) Give an example of groups \(G, H\) such that \(\text{Aut}(G)\) and \(\text{Aut}(H)\) are finite but \(\text{Aut}(G \times H)\) is infinite. (Here, \(\text{Aut}(G)\) denotes the set of automorphisms of \(G\).) No proof required.

   (b) Give an example of an ideal in \(\mathbb{Z}[x]\) that is prime but not maximal. No proof required.

   (c) Give an example of an integral domain that is not a unique factorization domain. No proof required.

   (d) State Eisenstein’s criterion for a polynomial \(f \in \mathbb{Z}[x]\) to be irreducible over \(\mathbb{Q}\). No proof required.

2. (10 pts.) Let \(V\) be the real vector space of functions on \(\mathbb{R}\) spanned by the set of real-valued functions \(\{e^x, xe^x, x^2 e^x, e^{2x}\}\). Let \(T : V \rightarrow V\) be the linear operator on \(V\) defined by \(T(f) = f'\). Find (i) a Jordan canonical form of \(T\), and (ii) a Jordan canonical basis.

3. (10 pts.) Let \(f : V \rightarrow V\) be an endomorphism of a finite-dimensional vector space \(V\). Show that there is a subspace \(U\) of \(V\) such that \(f(U) = f(V)\) and \(V = U \oplus \ker f\).

4. (10 pts.) Let \(G\) be a finitely generated abelian group. Prove that there are no nontrivial homomorphisms \(\phi : \mathbb{Q} \rightarrow G\), where \(\mathbb{Q}\) denotes the additive rationals.
5. (10 pts.) Let $G$ be a simple group of order $n$. Let $H$ be a subgroup of $G$ of index $k$ with $H \neq G$. Show that $n$ divides $k!$.

6. (10 pts.) Let $R$ be a commutative ring with unity 1. Suppose each subring of $R$ contains 1. Prove that $R$ is a field of nonzero characteristic.

7. (10 pts.) Let $R$ be a ring with 1, let $a \in R$, and suppose $a^n = 0$ for some $n \in \mathbb{N}$. Prove that $1 + a$ is a unit of $R$.

8. (10 pts.) Let $R$ be a commutative ring with 1. Let $m$ be a maximal ideal of $R$ such that $m \cdot m = 0$.

   (a) Prove that $m$ is the only maximal ideal of $R$.

   (b) Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be a polynomial in $R[x]$ such that $a_i \in m$ for all $i$ and $a_0 \neq 0$. Prove that $f(x)$ is irreducible, i.e., $f(x)$ is not a product of two polynomials in $R[x]$ of degree strictly smaller than $\deg f$.

9. (10 pts.) Let $F$ denote a finite field of order $2^5 = 32$. Prove that for each integer $1 \leq n < 32$ and each $a \in F$, the equation $x^n = a$ has a solution in $F$.

10. (10 pts.) Determine with proof the degree of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.
• Be sure to fully justify all answers.

• Notation The sets of integers, rational numbers, real numbers, and complex numbers are denoted \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \), respectively. All rings are understood to have a unit.

• Scoring Each single part problem is worth 10 points. Each part of a multiple part problem is worth 5 points. (e.g. Problem 1 is worth 10 points, Problem 2 is worth 25 points.)

• Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.

(1) Prove that for a group \( G \) and positive integer \( k \), if \( G \) contains an index \( k \) subgroup, then the intersection of all index \( k \) subgroups of \( G \) is a normal subgroup.

(2) Let \( F : G \to G \) be an endomorphism, that is, a homomorphism from the group \( G \) to itself. Let \( F^n \) denote the \( n \)-fold composition of \( F \) with itself, and let \( K_n = \text{Kernel}(F^n) \).

(a) Show that \( K_n \subseteq K_{n+1} \) for all \( n \).

(b) Let \( F : (\mathbb{Z}/16\mathbb{Z})^3 \to (\mathbb{Z}/16\mathbb{Z})^3 \) be the endomorphism defined by \( F(x, y, z) = (2z, 2x, 8y) \). For all \( n \geq 1 \), describe \( K_n \) as a direct sum of cyclic groups.

(c) Show that if \( F \) is an endomorphism of the symmetric group \( S_5 \), \( K_{n+1} = K_n \) for all \( n \geq 2 \).

(d) Give an example of an endomorphism \( F \) of the symmetric group \( S_5 \) for which \( K_2 \neq K_1 \).

(e) Prove that for general \( G \) and \( F \), if \( K_n = K_{n+1} \), then \( K_n = K_{n+i} \) for all \( i \geq 0 \).

(3) Let \( S = \{(x, y) \mid 23x + 31y = 1, x + y < 100\} \in \mathbb{Z}^2 \). Find the element of \( S \) for which \( x + y \) is as large as possible.

(4) Let \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) and \( S : \mathbb{R}^4 \to \mathbb{R}^1 \) be linear transformations given by:

\[
T(x, y, z) = (x + 2y + z, x - y + 4z, x - y + 4z, 2x + y + 5z)
\]

\[
S(x, y, z, w) = (x - y + 2z - w).
\]

Find two sets of vectors in \( \mathbb{R}^4 \), \( \{\alpha_1, \ldots, \alpha_m\} \) and \( \{\beta_1, \ldots, \beta_n\} \) such that \( \{\alpha_1, \ldots, \alpha_m\} \) is a basis of \( \text{Im}(T) \) and \( \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\} \) is a basis for \( \text{Ker}(S) \). Justify your answer.

(5) Let \( T_1, T_2 : \mathbb{C}^3 \to \mathbb{C}^3 \) be linear transformations. Show that if both linear transformations have minimal polynomials of degrees at most 2, then there is a vector that is an eigenvector for both \( T_1 \) and \( T_2 \).

(6) Let \( R \) be a commutative ring with unity and suppose that for every \( r \in R \) there is an \( n \geq 2 \) so that \( r^n = r \). Show that every prime ideal in \( R \) is maximal.

(7) Suppose that \( R \) is an integral domain. Is it possible that \( R \) contains additive subgroups isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{Z}/q\mathbb{Z} \) for \( p \) and \( q \) distinct primes? Justify your answer.
(8) Prove that the polynomial \(2x^4 + x + 1 \in \mathbb{Q}[x]\) is irreducible. Justify all your work.

(9) Let \(F_q\) denote the finite field with \(q\) elements. Show that for any \(a \in F_q\) the equation \(x^n = a\) has a solution in \(F_q\) if \(n\) is relatively prime to \(q - 1\).

(10) Let \(p(t) = t^3 - 2 \in \mathbb{Q}[t]\). Let \(\alpha = \sqrt[3]{2}\) be the real root of \(p\) and let \(\beta\) be a complex root of \(p\). Determine if \(\alpha \in \mathbb{Q}[\beta]\) and explain your answer.

(11) Let \(d > 1\) and let \(p(x)\) and \(q(x)\) be relatively prime irreducible polynomials in \(\mathbb{Q}[x]\) of degree \(d\). Suppose \(p(\alpha) = 0 = q(\beta)\) for some \(\alpha, \beta \in \mathbb{C}\). It follows that \(1 \leq [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] \leq d\).

(a) Find an example of a \(d, p, q, \alpha,\) and \(\beta\), so that \([\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 1\).

(b) Find an example of a \(d, p, q, \alpha,\) and \(\beta\), so that \([\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = d\).
Algebra Tier 1
January 2008

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page.

Problem 1. Find eigenvalues and the corresponding eigenvectors of the complex matrix

\[
A = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Problem 2. Let \( A \) be a 5 \times 5 complex matrix such that \( A^3 = 0 \). List all possible Jordan canonical forms of \( A \).

Problem 3. Find a 5 \times 5 matrix \( A \) with rational entries whose minimal polynomial is \((x^3 + 1)(x + 2)^2\).

Problem 4. Let \( A \) be a complex \( n \times n \) matrix such that \( A^m = I \) for some \( m \geq 1 \). Prove that \( A \) is conjugate to a diagonal matrix.

Problem 5. Consider the group \( \mathbb{R}, + \), the additive group of the real numbers.
   a) Show that any homomorphism from a finite group to \( \mathbb{R}, + \) has to be the trivial homomorphism.
   b) Show that any homomorphism from \( \mathbb{R}, + \) to a finite group has to be the trivial homomorphism.

Problem 6. Consider the subgroup \( H \) of the group \( \mathbb{Z}/12 \times \mathbb{Z}/12 \) generated by the element \((a^4, a^6)\), where \( a \) is a generator of \( \mathbb{Z}/12 \).
   a) What is the order of \( H \)? List its elements.
   b) How many elements are there in \((\mathbb{Z}/12 \times \mathbb{Z}/12)/H\)?
   c) Write \((\mathbb{Z}/12 \times \mathbb{Z}/12)/H\) as a product of cyclic groups, each of which has order equal to a power of some prime. Find a generator for each of these cyclic subgroups.

Problem 7. Show that in a finite group of odd order every element is a square.

Problem 8. For each of the following subgroups of \( S_4 \) (the permutation group on four elements), say what its order is and justify your answer.
   a) The subgroup generated by \((1, 2)\) and \((3, 4)\).
   b) The subgroup generated by \((1, 2), (3, 4), \) and \((1, 3)\).
   c) The subgroup generated by \((1, 2), (3, 4), \) and \((1, 3)(2, 4)\).
   d) The subgroup generated by \((1, 2)\) and \((1, 3)\).

Problem 9. Let \( R \) be an integral domain that contains a field \( K \). Show that if \( R \) is a finite dimensional vector space over \( K \), then \( R \) is a field.

Problem 10. Let \( f(x) \) be a polynomial with coefficients from a finite field \( F \) with \( q \) elements. Show that if \( f(x) \) has no roots in \( F \), then \( f(x) \) and \( x^q - x \) are relatively prime.

Problem 11. Let \( \alpha \) be a root of an irreducible polynomial \( x^3 - 2x + 2 \) over \( \mathbb{Q} \). Find the multiplicative inverse of \( \alpha^2 + \alpha + 1 \) in \( \mathbb{Q}[\alpha] \) in the form \( a + b\alpha + c\alpha^2 \) with \( a, b, c \in \mathbb{Q} \).

Problem 12. Let \( f(x) \) and \( g(x) \) be irreducible polynomials over \( \mathbb{Q}[x] \). Let \( \alpha \) be a root of \( f(x) \) and let \( \beta \) be a root of \( g(x) \). Show that \( f(x) \) is irreducible over \( \mathbb{Q}(\beta) \) if and only if \( g(x) \) is irreducible over \( \mathbb{Q}(\alpha) \).
(9) 1. Complete the following definitions:

(a) Let $G$ be a group and let $g \in G$. The order of $g$ is

(b) Let $K/F$ be a field extension. An element $a \in K$ is called algebraic if

(c) An ideal $I$ in a commutative ring $R$ is called prime if

(12) 2. Let $G$ be a group and let $G_2 = \{ g^2 \mid g \in G \}$. Let $H$ denote the intersection of all subgroups of $G$ containing $G_2$.

(a) Prove that $H$ is a normal subgroup of $G$.

(b) Prove that $G/H$ is abelian.

(c) Prove that if $G/H$ is finite, its order is a power of 2.

(10) 3. Let $K$ be a field. Let $a, b \in K$ and let $R = K[x]/(x^2 + ax + b)$. Prove that exactly one of the following is true:

- $R$ is a field.
- $R$ is isomorphic to $K^2$, the direct sum of two copies of $K$.
- There is a nonzero element $r \in R$ such that $r^2 = 0$.

(8) 4. A complex matrix $A$ has characteristic polynomial $(x - 2)^4(x + 2)$ and minimal polynomial $(x - 2)(x + 2)$. Determine the possible Jordan canonical forms for $A$.

(12) 5. Let $V$ be an $n$-dimensional real vector space.

(a) Let $a, b$ be nonnegative integers. Prove there are subspaces $V_a$ and $V_b$ of dimension $a, b$ respectively with $V_a \cap V_b = 0$ if and only if $a + b \leq n$.

(b) Let $a, b, c$ be nonnegative integers. Prove there are subspaces $V_a, V_b$ and $V_c$ of dimension $a, b, c$ respectively with $V_a \cap V_b \cap V_c = 0$ if and only if $a \leq n, b \leq n, c \leq n$ and $a + b + c \leq 2n$.

(8) 6. Let $F$ be a field. Determine the possible finite groups $G$ that are isomorphic to a subgroup of $F^+$, the additive group of $F$. 

1
(10)7. A nonzero prime ideal $P$ in a commutative ring $R$ is called minimal if the only nonzero prime ideal $Q$ contained in $P$ is $P$ itself. Now let $F$ be a field and let $R = F[x, y]$, the polynomial ring in two variables over $F$. Prove that if $P$ is a minimal prime ideal of $R$ there is an irreducible element $f(x, y)$ in $R$ such that $P = (f(x, y))$.

(10)8. Let $D_n$ denote the dihedral group of order $2n$ (that is, $D_n$ is the group of symmetries of the regular $n$-gon). Let $G$ be a finite group. Prove that if there is a nontrivial homomorphism from $D_n$ to $G$ then the order of $G$ is even.

(10)9. Let $G$ be a group and let $M, N$ be normal subgroups such that $MN = G$ and $M \cap N = \{e\}$. Prove that $G$ is isomorphic to the direct product $G/M \times G/N$.

(10)10. Let $M_n(\mathbb{Q})$ denote the ring of $n \times n$ matrices over the rationals. Let $K$ be a subring of $M_n(\mathbb{Q})$ such that $K$ is a field and $K$ contains $\mathbb{Q}$. Prove that the degree $[K : \mathbb{Q}]$ is finite and $[K : \mathbb{Q}]$ divides $n$. 
Instructions: Each problem or major part of a problem counts five points, as indicated, for a total of 90 points. Start each problem on a new sheet of paper. Unless otherwise stated you should show your work and justify your claims.

Problem 1 (10 points).
(1) Prove that every subgroup of a cyclic group is cyclic.
(2) Determine, up to isomorphism, the finitely generated abelian groups with the property that every proper subgroup is cyclic.

Problem 2 (5 points). Let $G$ be a group. Define a subgroup $H \leq G$ to be characteristic if for every isomorphism $\varphi : G \rightarrow G$ one has $\varphi(H) \subseteq H$. Now suppose that $H \leq G$ is a normal subgroup and $K \leq H$ is a characteristic subgroup of $H$. Prove that $K$ is a normal subgroup of $G$.

Problem 3 (5 points). Let $G$ be a finite abelian group of order $n$, in which the group operation is written multiplicatively. Suppose that the map $f : x \mapsto x^m$ is an automorphism of $G$, for some positive integer $m$. Prove that $\gcd(n, m) = 1$.

Problem 4 (10 points). Let $V$ denote a finite-dimensional complex vector space, and $L(V)$ the complex vector space of linear transformations from $V$ to itself. Given $A \in L(V)$, the function $T_A : L(V) \rightarrow L(V)$ defined by $T_A(X) = AX -XA$ for all $X \in L(V)$ is a linear transformation.
(1) Suppose that $A, B \in L(V)$ have the same Jordan canonical form. Prove that $T_A$ and $T_B$ have the same Jordan canonical form.
(2) Suppose that the dimension of $V$ is equal to 2. Prove that for every $A \in L(V)$, the rank of the transformation $T_A$ is either 0 or 2.

Problem 5 (5 points). Let $V$ be a real vector space and $T : V \rightarrow V$ a linear transformation. Suppose every nonzero vector in $V$ is an eigenvector of $T$. Prove that $T$ is a scalar multiple of the identity.
Problem 6 (10 points).
(1) Let \( A \) be a \( 2 \times 2 \) matrix with entries in the field of real numbers such that
\[
A^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Show that \( A \) is similar over the real numbers to the matrix
\[
B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
(2) Let \( A \) be an \( n \times n \) matrix with entries in the field of real numbers such that
\( A^2 + I_n = 0 \) where \( I_n \) is the \( n \times n \) identity matrix. Show that \( n \) is even and \( A \) is similar over the real numbers to the matrix
\[
B = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix},
\]
where \( n = 2m \).

Problem 7 (10 points). Let \( R \) be a commutative ring with 1 that contains exactly three ideals.
(1) Show that every nonzero element of \( R \) is either a unit or a zero divisor.
(2) Is the converse true? Justify your answer.

Problem 8 (5 points). For a prime integer \( p \), let \( F_p \) denote the field of \( p \) elements. Suppose the greatest common divisor of the polynomials \( f(x) = 6x^3 + 10x^2 - 10x + 16 \) and \( g(x) = 6x^2 + 10x - 16 \) in \( F_p[x] \) is 1. Find \( p \).

Problem 9 (10 points). Let \( F_3 \) denote the finite field with 3 elements and \( \overline{F}_3 \) be its algebraic closure. Let \( K \) be the splitting field of \( g(x) = x^{21} - 1 \).
(1) Find the number of zeros of \( g(x) \) in the field \( \overline{F}_3 \).
(2) (a) Find the number of elements in \( K \). (b) What are the numbers of elements in the maximal proper subfields of \( K \)? (A subfield of \( K \) is said to be proper if it is not equal to \( K \) itself.)

Problem 10 (10 points).
(1) Suppose \( \gamma \) is a complex number such that \( \gamma^2 \) is algebraic over \( \mathbb{Q} \). Show that \( \gamma \) is algebraic over \( \mathbb{Q} \).
(2) Let \( \alpha, \beta \) be complex numbers such that \( \alpha \) is transcendental over \( \mathbb{Q} \). Show that at least one of \( \alpha - \beta \) and \( \alpha \beta \) is transcendental.

Problem 11 (10 points). Let \( D \) be a domain. Two non-zero ideals \( I, J \subset D \) are called “comaximal” if \( I + J = D \) and the two ideals are called “coprime” if \( I \cap J = I \cdot J \).
(1) Show that if the ideals \( I, J \subset D \) are comaximal, then they are coprime.
(2) Show that if \( D \) is a principal ideal domain and the ideals \( I, J \subset D \) are coprime, then they are comaximal.
Tier I Algebra Exam
August, 2009

- Be sure to fully justify all answers.
- **Notation** The sets of integers, rational numbers, and real numbers are denoted \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{R} \), respectively. For a prime integer \( p \), \( \mathbb{Z}/p \mathbb{Z} \) denotes the quotient \( \mathbb{Z}/p \mathbb{Z} \). All rings are understood to have a unit.
- **Scoring** Each problem is worth 10 points.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.

(1) Let \( A \) and \( B \) be finite subgroups of a group \( G \) of relatively prime orders; that is, \( \gcd(|A|, |B|) = 1 \). Prove that the function \( \phi : A \times B \to G \) defined by \( \phi(a, b) = ab \) is injective.

(2) Find an element of largest order in the symmetric group \( S_{12} \). Justify your answer.

(3) (a) Let \( G \) and \( H \) be abelian groups and let \( \text{Hom}(G, H) \) denote the set of all group homomorphisms from \( G \) to \( H \). If \( \phi, \psi \in \text{Hom}(G, H) \), define \( \phi + \psi \) by \( (\phi + \psi)(g) = \phi(g) + \psi(g) \) for all \( g \in G \). Prove that \( \text{Hom}(G, H) \) is an abelian group.

(b) Let \( C \) be a cyclic group such that \( \text{Hom}(C, \mathbb{Z}/p \mathbb{Z}) \cong \mathbb{Z}/p \mathbb{Z} \). What can you say about the order of \( C \)?

(c) Let \( G \) be a finitely generated abelian group and \( \text{Hom}(G, \mathbb{Z}/p \mathbb{Z}) \cong (\mathbb{Z}/p \mathbb{Z})^n \). What does this tell you about \( G \)?

(4) Let \( A, B \) be commuting \( 2 \times 2 \) real matrices with characteristic polynomials \( x^2 - 3x + 2 \) and \( x^2 - 1 \), respectively. Show that either \( A + B \) or \( A - B \) has determinant 0.

(5) Suppose that \( T \) is a linear transformation of a finite dimensional real vector space \( V \) having characteristic polynomial \( f(t)g(t) \) where \( f \) and \( g \) are relatively prime. Show that \( V = \text{Ker}(f(T)) \oplus \text{Ker}(g(T)) \).

(6) Let \( T \) be a linear transformation of a finite dimensional real vector space \( V \) and assume that \( V \) is spanned by eigenvectors of \( T \). If \( T(W) \subset W \) for some subspace \( W \subset V \), show that \( W \) is spanned by eigenvectors. (Hint: consider the minimal polynomial of \( T \).)
(7) Let $F$ denote the field with two elements and let $E$ be an extension field of $F$.

(a) Show that if $\alpha \in E$ satisfies $f(\alpha) = 0$ for some $f \in F[x]$, then $f(\alpha^2) = 0$.

(b) Suppose that $\alpha \in E$ is a root of the polynomial $f(x) = x^5 + x^2 + 1 \in F[x]$. List all roots of $f(x)$ in $E$ in the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$ with each $a_i = 0$ or $1$.

(8) Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be irreducible rational polynomials, having roots $\alpha$ and $\beta$, respectively. Find necessary and sufficient conditions on the coefficients $(a, b, c, d)$ that imply that $\mathbb{Q}(\alpha)$ is isomorphic to $\mathbb{Q}(\beta)$. Prove that your conditions are both necessary and sufficient.

(9) (a) Is the ring $\mathbb{Z}[i]$ of Gaussian integers an integral domain? Justify your answer.

(b) Let $R = \mathbb{Z}[T]/\langle T^4 - 1 \rangle$, where $\langle T^4 - 1 \rangle$ is the ideal generated by $T^4 - 1$. Is $R$ an integral domain? Justify your answer.

(c) Show that sending $T$ to $i$ determines a ring homomorphism $\psi : R \to \mathbb{Z}[i]$. Describe $\text{Ker} \psi$ as all elements $a + bT + cT^2 + dT^3 \in R$ where $a, b, c, d$ satisfy some conditions.

(10) Let $F^*$ denote the multiplicative group of all nonzero elements of a finite field $F$. Show that $F^*$ is cyclic.
All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ stands for integers, rational numbers and complex numbers respectively.

**Problem 1.** Let $A$ be a $n \times n$ complex matrix which does not have eigenvalue $-1$. Show that the matrix $A + I_n$ is invertible. ($I_n$ is the identity $n \times n$ matrix.)

**Problem 2.**
(a) Find the eigenvalues of the complex matrix

$$A = \begin{bmatrix}
0 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0
\end{bmatrix}$$

(b) Find the eigenvectors of $A$.
(c) Find an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

**Problem 3.** Let $A, B$ be $n \times n$ complex matrices such that $AB = BA$. Prove that there exists a vector $v \neq 0$ in $\mathbb{C}^n$ which is an eigenvector for $A$ and for $B$.

**Problem 4.** Suppose that $G$ is a group of order 60 that has 5 conjugacy classes of orders 1,15,20,12,12. Prove that $G$ is a simple group.

**Problem 5.** Prove that any group of order 49 is abelian.

**Problem 6.** How many conjugacy classes are there in the symmetric group $S_5$?

**Problem 7.** Let $G = GL_2(\mathbb{F}_5)$, the group of invertible $2 \times 2$ matrices with entries in the field $\mathbb{F}_5$ with 5 elements. What is the order of $G$?

**Problem 8.** Let $G$ and $H$ be any pair of groups and let $S = \text{Hom}(G, H)$ denote the set of homomorphisms from $G$ to $H$.

a) Prove that if $H$ is an abelian group, then the operation “+” on $S$ given by $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ makes $S$ into an abelian group.

b) Prove that if $G$ is a finite cyclic group, then $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is isomorphic to $G$.

c) Find an infinite abelian group $G$ so that $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is not isomorphic to $G$.

**Problem 9.** Describe the prime ideals in the ring $\mathbb{C}[x]$.

**Problem 10.** Find the degree of the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

**Problem 11.**
(a) Prove that the polynomial $x^2 + x + 1$ is irreducible over the field $\mathbb{F}_2$ with two elements.

b) Factor $x^9 - x$ into irreducible polynomials in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field with three elements.

**Problem 12.** Determine the following ideals in $\mathbb{Z}$ by giving generators:

$$ (2) + (3), \quad (4) + (6), \quad (2) \cap (3), \quad (4) \cap (6) $$

**Problem 13.** Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $n$ such that $f$ and $f'$ (the derivative of $f$) have no common roots. Show that the quotient ring $\mathbb{C}[x]/(f)$ is isomorphic to $\mathbb{C} \times \ldots \times \mathbb{C}$ ($n$ times).
1. Find all irreducible monic quadratic polynomials in $\mathbb{Z}_3[x]$. 
(Monic: coefficient of the highest power of $x$ is one.)

2. Let $G$ be a finite group and $\Phi : G \rightarrow G$ an automorphism.

   (a) Show that $\Phi$ maps a conjugacy class of $G$ into a conjugacy class of $G$.

   (b) Give a concrete example of non-trivial $G$ and $\Phi$ such that $\{e\}$ is the only conjugacy class of $G$ that $\Phi$ maps into itself. Explain.

   (c) Show that if $G = S_5$ (the symmetric group on five letters), then $g$ and $\Phi(g)$ must be conjugate for any $g \in G$.

3. Let $V$ and $W$ be real vector spaces, and let $T : V \rightarrow W$ be a linear map. If the dimensions of $V$ and $W$ are 3 and 5, respectively, then for any bases $B$ of $V$ and $B'$ of $W$, we can represent $T$ by a $5 \times 3$ matrix $A_{T,B,B'}$. Find a set $S$ of $5 \times 3$ matrices as small as possible such that for any $T : V \rightarrow W$ there are bases $B$ of $V$ and $B'$ of $W$ such that $A_{T,B,B'} \in S$.

4. Is it possible to find a field $F$ with at most 100 elements so that $F$ has exactly five different proper subfields? If so, find all such fields. If not, prove that no such field $F$ exists.

5. Let $G$ be the group of rigid motions (more specifically, rotations) in $\mathbb{R}^3$ generated by $x = a 90^\circ$ degree rotation about the $x$-axis, and $y = a 90^\circ$ degree rotation about the $y$-axis.

   (a) How many elements does $G$ have?

   (b) Show that the subgroup generated by $x^2$ and $y^2$ is a normal subgroup of $G$.

6. In this problem, $R$ is a finite commutative ring with identity. Define $a \in R$ to be periodic of period $k$ if $a$, $a^2$, $\ldots$, $a^k$ are all different, but $a^{k+1} = a$.

   (a) In $R = \mathbb{Z}_{76}$, find an element $a \neq 0, 1$ of period 1.
(b) In the same ring $R = \mathbb{Z}_{76}$ find an element that is not periodic.

(c) In $R = \mathbb{Z}_{76}$, list the possible periods and the number of elements of each period.

7. In this problem, $R$ is a finite commutative ring with identity. Let $p(x) \in R[x]$, the ring of polynomials over $R$.

(a) Show that $a \in R$ is a root of $p(x)$ if and only if $p(x)$ can be written as $p(x) = (x - a)g(x)$ with $g(x) \in R[x]$ of degree one less than the degree of $g(x)$.

(b) Prove or give a counterexample: A polynomial of $p(x) \in R[x]$ of degree $n$ can have at most $n$ distinct roots in $R$.

8. Consider $S_5$, the symmetric group on 5 letters. If $\sigma \in S_5$ has order 6, how many elements of $S_5$ commute with $\sigma$?

9. Let $A$ be a $5 \times 5$ real matrix of rank 2 having $\lambda = -i$ as one of its eigenvalues. Show that $A^3 = -A$ and that $A$ is diagonalizable (as a complex matrix).

10. (a) Give an example of an irreducible monic polynomial of degree 4 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$. Explain why your example has the stated properties.

(b) Show that there is no irreducible monic polynomial of degree 5 in $\mathbb{Z}[x]$ that is reducible in the field $\mathbb{Q}[\sqrt{2}]$.

11. Let $M$ be the ring of $3 \times 3$ matrices with integer entries. Find all maximal two-sided ideals of $M$.

12. For which values of $n$ in $\mathbb{Z}$ does the ring $\mathbb{Z}[x]/(x^3 + nx + 3)$ have no zero divisors?
1. For an element $g$ of a group $G$, define the centralizer subgroup $C(g) = \{h \in G \mid hg = gh\}$. What is the minimal order of the centralizer subgroup of an element of order 2 in $S_6$? Explain.

2. Let $G$ be a group and $H_3$ and $H_5$ normal subgroups of $G$ of index 3 and 5 respectively. Prove that every element $g \in G$ can be written in the form $g = h_3h_5$, with $h_3 \in H_3$ and $h_5 \in H_5$.

3. Show that every finite group whose order is at least 3 has a non-trivial automorphism.

4. The following matrix has four distinct real eigenvalues. Find their sum and their product.

   $$\begin{pmatrix}
   1 & 0 & 0 & 0 \\
   3 & 0 & 2 & 1 \\
   0 & 1 & 0 & 3 \\
   0 & 0 & 3 & 2
   \end{pmatrix}$$

5. Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation $T(a, b, c, d) = (a + b - c, c + d)$. Find a basis for the null space.

6. A $5 \times 5$ matrix $A$ satisfies the equation $(A-2I)^3(A+2I)^2 = 0$. Assume that there are at least two linearly independent vectors $v$ that satisfy $Av = 2v$. What are the possibilities for the Jordan canonical form? List only one in each conjugacy class.

7. Prove that in a commutative ring with a finite number of elements, prime ideals are maximal.
8. Let $F_4$ be the finite field with four elements. Express

$$F_4[x]/\langle x^4 + x^3 + x^2 + 1 \rangle$$

as a product of fields. Prove your result.

9. Recall an element $r$ of a ring $R$ is a **unit** if there is an $s \in R$ so that $rs = 1 = sr$ and an element $r$ of a ring $R$ is **nilpotent** if there is a positive integer $n$ so that $r^n = 0$.

   (a) Give an example of a ring $R$ and a unit $r \in R$ with $r \neq 1$.

   (b) Give an example of a ring $R$ and a nilpotent element $r \in R$ with $r \neq 0$.

   (c) Show that for any ring $R$ and for any element $r \in R$, that $r$ is a nilpotent element of $R$ if and only if $1 - rx$ is a unit in the polynomial ring $R[x]$.

10. Let $M_n(\mathbb{C})$ denote the vector space over $\mathbb{C}$ of all $n \times n$ complex matrices. Prove that if $M$ is a complex $n \times n$ matrix, then $C(M) = \{ A \in M_n(\mathbb{C}) \mid AM = MA \}$ is a subspace of $M_n(\mathbb{C})$ of dimension at least $n$. 

2
1. (8 points) Let $A$ be a matrix in $GL_n(\mathbb{C})$. Show that if $A$ has finite order (i.e., $A^k$ is the identity matrix for some $k \geq 1$), then $A$ is diagonalizable.

2. (8 points) Let $V$ be a finite-dimensional real vector space of dimension $n$. Define an equivalence relation $\sim$ on the set $\text{End}_\mathbb{R}(V)$ of $\mathbb{R}$-linear homomorphisms $V \to V$ as follows: if $S, T \in \text{End}_\mathbb{R}(V)$ then $S \sim T$ if and only if there are invertible maps $A : V \to V$ and $B : V \to V$ such that $S = BTA$. (You may assume this is an equivalence relation.)

Determine, as a function of $n$, the number of equivalence classes.

3. (8 points) Let $n \geq 2$. Let $A$ be the $n$-by-$n$ matrix with zeros on the diagonal and ones everywhere else. Find the characteristic polynomial of $A$.

4. (8 points) Find the Jordan canonical form of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}$.

Justify your answer.

5. (8 points) Let $R = K[x, y, z]/(x^2 - yz)$, where $K$ is a field. Show that $R$ is an integral domain, but not a unique factorization domain.

6. (8 points) Let $P$ be a prime ideal in a commutative ring $R$ with 1, and let $f(x) \in R[x]$ be a polynomial of positive degree. Prove the following statement: if all but the leading coefficient of $f(x)$ are in $P$ and $f(x) = g(x)h(x)$, for some non-constant polynomials $g(x), h(x) \in R[x]$, then the constant term of $f(x)$ is in $P^2$.

[We recall that $P^2$ is the ideal generated by all elements of the form $ab$, where $a, b \in P$.]
7. (10 points) Let \( p \) be a prime number and denote by \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) the field with \( p \) elements. For a positive integer \( n \) let \( \mathbb{F}_{p^n} \) be the splitting field of \( x^{p^n} - x \in \mathbb{F}_p[x] \). Prove that the following statements are equivalent:

1) \( k | n \).
2) \( (p^k - 1)|(p^n - 1) \).
3) \( \mathbb{F}_{p^k} \subset \mathbb{F}_{p^n} \).

8. (10 points) i) Show that \( x^3 - 2 \) and \( x^5 - 2 \) are irreducible over \( \mathbb{Q} \).

ii) How many field homomorphisms are there from \( \mathbb{Q}[\sqrt[3]{2}, \sqrt[5]{2}] \) to \( \mathbb{C} \)?

iii) Prove that the degree of \( \sqrt[3]{2} + \sqrt[5]{2} \) over \( \mathbb{Q} \) is 15.

9. (8 points) Let \( p \) be a prime number. Prove that any group of order \( p^2 \) is abelian.

10. (8 points) Let \( a \) be an element of a group \( G \). Prove that \( a \) commutes with each of its conjugates in \( G \) if and only if \( a \) belongs to an abelian normal subgroup of \( G \).

11. (8 points) Find the cardinality of \( \text{Hom}(\mathbb{Z}/20\mathbb{Z}, \mathbb{Z}/50\mathbb{Z}) \), where \( \text{Hom}(\cdot, \cdot) \) denotes the set of group homomorphisms.

12. (8 points) Let \( G \) be a finite group, and let \( M \subset G \) be a maximal subgroup, i.e., \( M \) is a proper subgroup of \( G \) and there is no subgroup \( M' \) such that \( M \subset M' \subset G \). Show that if \( M \) is a normal subgroup of \( G \) then \( |G : M| \) is prime.

[Hint. Consider the homomorphism \( G \to G/M \).]
Algebra Tier 1
January 2012

All your answers should be justified. A correct answer without a correct proof earns little credit. All questions are worth the same number of points. Write a solution of each problem on a separate page. The notation $F_n, R, Z$ stands for the field with $n$ elements, the field of real numbers, and the ring of integers respectively.

**Problem 1.** Find the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}$$

**Problem 2.** Find the matrix $A^{2011}$, where $A$ is the matrix from problem 1 above.

**Problem 3.** Find the eigenvalues and a basis for the eigenspaces of the matrix

$$B = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

**Problem 4.** Find the matrix $e^C := I + C + \frac{C^2}{2!} + \frac{C^3}{3!} + ..., \text{ where}$

$$C = \begin{bmatrix}
1 & 4 \\
1 & 1
\end{bmatrix}$$

**Problem 5.** Let $a$ and $b$ be elements of a group $G$. Prove that $ab$ and $ba$ have the same order.

**Problem 6.** Prove that if $G$ is a nonabelian group, then $G/Z(G)$ is not cyclic. Here $Z(G)$ denotes the center of $G$.

**Problem 7.** Prove that if $G$ is a finite nonabelian group of order $p^3$, where $p$ is a prime, then $Z(G) = [G, G]$, where $Z(G)$ denotes the center of $G$ and $[G, G]$ denotes the commutator subgroup of $G$.

**Problem 8.** Let $C_2$ denote a cyclic group of order 2. Determine the group $Aut(C_2 \times C_2)$, calculating its order and identifying it with a familiar group.

**Problem 9.** Find all irreducible polynomials of degree $\leq 4$ in $F_2[x]$.

**Problem 10.** Find the set of polynomials in $F_2[x]$ which are the minimal polynomials of elements in $F_{16}$.

**Problem 11.** Prove that the rings $F_{16}$, $F_4 \times F_4$, and $Z/16Z$ are pairwise non-isomorphic.

**Problem 12.** Find all the maximal ideals in the ring $R[x]$.

**Problem 13.** Let $R$ be the ring of Gaussian integers and $I \subset R$ be an ideal. If $R/I$ has 4 elements what are the possibilities for $I$ and $R/I$?
(1) Let $f: \mathbb{Z}^3 \to \mathbb{Z}^4$ be the function
\[ f(a, b, c) = (a + b + c, a + 3b + c, a + b + 5c, 4a + 8b) \]
(a) Prove that $f$ is a group homomorphism.
(b) Let $H$ denote the image of $f$. Find an element of infinite order in $\mathbb{Z}^4/H$.
(c) Calculate the order of the torsion subgroup of $\mathbb{Z}^4/H$.

(2) Let $A$ be the matrix
\[ A = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \]
Calculate $[K : \mathbb{Q}]$ where $K$ is the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and the eigenvalues of $A$.

(3) Let $R = \mathbb{Z}[x]/I$, where $I$ is the ideal generated by $x^2 - 5x - 2$.
Let $S$ denote the ring of $2 \times 2$ integer matrices: $S = M_2(\mathbb{Z})$.
Let $B$ denote the matrix
\[ B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \]
(a) Show that there exists a unique ring homomorphism $f: R \to S$ satisfying $f(x + I) = B$.
(b) Let $J$ denote the ideal in $R$ generated by the element $(x - 1) + I$. Is $f(J)$ an ideal in $S$? Why or why not?

(4) Does there exist a non-abelian group of order 2012?

(5) Let $n \in \{2, 3, 7\}$ and consider the ring
\[ R_n = (\mathbb{Z}/n)[x]/(x^3 + x^2 + x + 2) \]
For which $n \in \{2, 3, 7\}$ (if any) is $R_n$ a field? For which $n$ (if any) is $R_n$ a integral domain but not a field? For which $n$ (if any) is $R_n$ not an integral domain? Justify all your conclusions.
(6) Let $R$ be the subring of $\mathbb{R}$ given by $R = \{n + m\sqrt{-10} \mid m, n \in \mathbb{Z}\}$. Show that the element $2 - \sqrt{-10}$ is irreducible in $R$ but not prime.

(7) Let $G$ be a finite abelian group and let $\phi : G \to G$ be a group homomorphism. Note that for all positive integers $k$ the function $\phi^k = \phi \circ \phi \circ \cdots \circ \phi$ is also a homomorphism from $G$ to $G$. Prove there is a positive integer $n$ such that $G \cong ker(\phi^n) \times \phi^n(G)$.

(8) Let $G$ be a group containing normal subgroups of order 3 and 5. Prove $G$ contains an element of order 15.

(9) Let $M$ be the following matrix:

$$
\begin{pmatrix}
1 & -1 & 2 \\
2 & -1 & 1 \\
-4 & 1 & 0 \\
3 & -2 & 3
\end{pmatrix}
$$

Prove or disprove (by giving a counterexample) each of the following statements:

(a) For every $3 \times 4$ complex matrix $N$ there is a nonzero vector $v \in \mathbb{C}^4$ such that $MNv = 0$.

(b) For every $3 \times 4$ complex matrix $N$ there is a nonzero vector $v \in \mathbb{C}^3$ such that $NMv = 0$.

(10) Suppose that $K/F$ is a finite extension of fields and $p$ is the smallest prime dividing $[K : F]$. Prove that for all $\alpha \in K$, $F(\alpha) = F(\alpha^{p-1})$. 
Tier I Algebra Exam
January, 2013

• Be sure to fully justify all answers.

• Notation The sets of integers, rational numbers, real numbers, and complex numbers are denoted \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C}, \) respectively. All rings are understood to have a unit and ring homomorphisms to be unit preserving.

• Scoring Each problem is worth 10 points.

• Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number and your test number on each sheet of paper.

1. Give examples with brief justification:
   (a) A commutative ring with exactly one non-zero prime ideal.
   (b) A commutative ring with a non-zero prime ideal that is not maximal.
   (c) A UFD that is not a PID.
   (d) A \( 2 \times 2 \) integer matrix having \( 1 + \sqrt{2} \) as an eigenvalue.
   (e) A polynomial of degree 4 with integer coefficients that is irreducible over the rational numbers but not irreducible when reduced mod 3, mod 5, and mod 7.

2. Let \( R \) be a commutative ring with unit and let \( P < R \) be a prime ideal. Show that if \( R/P \) is a finite set, then \( P \) is a maximal ideal.

3. Find the degree of the field \( \mathbb{Q}(\sqrt{2}) \) as an extension of the field \( \mathbb{Q}(\sqrt{2}) \).

4. Let \( \mathbb{F} \) be a field with 8 elements and \( \mathbb{E} \) a field with 32 elements. Construct a (unit preserving) homomorphism of rings \( \mathbb{F} \to \mathbb{E} \) or prove that one cannot exist.

5. In \( \mathbb{R}^5 \), consider the subspaces
   \[ V = \langle (1, 2, 3, 3, 2), (0, 1, 0, 1, 1) \rangle \quad \text{and} \quad W = \langle (0, -1, 3, 2, -1), (1, 1, 0, -1, 1) \rangle, \]
   where \( \langle \rangle \) indicates span. Find a basis for \( V \cap W \).

6. Compute \( \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}^{100} \).

Test continues on other side.
7. Consider the matrix
\[
M = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
For which \( n \in \mathbb{Z} \) does there exist a matrix \( P \) (with entries in \( \mathbb{C} \)) such that \( P^n = M \)?

8. Suppose that \( \phi \) is a homomorphism from \( \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \) to itself satisfying \( \phi^5 = \text{id} \) (where \( \phi^5 = \phi \circ \phi \circ \phi \circ \phi \circ \phi \)). Show that \( \phi \) is the identity.

9. Consider the quotient of additive abelian groups \( G = \mathbb{Q}/\mathbb{Z} \). Prove that every finite subgroup of \( G \) is cyclic.

10. Consider the order 2 subgroup \( H = \{(1), (1 2)(3 4)\} \) of the symmetric group \( S_4 \).
   (a) What is the normalizer \( N(H) \)?
   (b) What numbers occur as orders of non-identity elements of the quotient group \( N(H)/H \)?

11. Classify up to isomorphism all groups with 38 elements: Give a list of non-isomorphic groups with 38 elements such that every group with 38 elements is isomorphic to one in your list. Be sure to justify that your list consists of non-isomorphic groups and that you have identified all groups with 38 elements up to isomorphism.

12. For an abelian group \( A \) and a positive integer \( n \), consider the automorphism of \( A \) given by multiplication by \( n \). Denote by \( _nA \) and \( A/n \) its kernel and cokernel (quotient), respectively. Let \( \phi: A \to B \) be a homomorphism of finite abelian groups, and assume that for all prime numbers \( p \), \( \phi \) induces an isomorphism \( _pA \to _pB \) and an isomorphism \( A/p \to B/p \). Show that \( \phi \) is an isomorphism.
1. (8 points) Prove Fermat’s Little Theorem: If \( p \) is a prime number and \( a \) is any integer, then \( a^p - a \) is divisible by \( p \).

2. (8 points) Compute \( A^{2013} \), where \( A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \).

3. (8 points) Construct an explicit isomorphism between \( \mathbb{Z}[\sqrt{-1}]/(7) \) and \( \mathbb{Z}[\sqrt{-2}]/(7) \), where \((7)\) denotes the ideal generated by \(7\).

4. (9 points) Let \( \mathbb{Z}/n\mathbb{Z} \) be the cyclic group of order \( n > 1 \).
   
   (a) Show that the automorphism group \( A = Aut(\mathbb{Z}/n\mathbb{Z}) \) is abelian.
   
   (b) What is the order of the automorphism group of the finite group \( G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z} \)?
   
   (c) Let \( G \) be as in part (b). Is the group \( Aut(G) \) abelian?

5. (8 points) Show that \( x^5 - (3 + i)x + 2 \) is irreducible in \((\mathbb{Z}[i])[x]\).

6. (8 points) For any pair of real numbers \( a \) and \( b \), let \( M_{a,b} \) be the \( n \times n \) matrix

\[
M_{a,b} = \begin{pmatrix}
    a & b & \ldots & b \\
    b & a & \ldots & b \\
    \vdots & \vdots & \ddots & \vdots \\
    b & b & \ldots & a
\end{pmatrix}
\]

with entries \( a \) on the diagonal and \( b \) off the diagonal. Find the eigenvalues of \( M_{a,b} \) and their multiplicities.

7. (10 points) Classify (up to isomorphism) all groups of order 8. (You may use the following fact without proof: if \( g^2 = 1 \) for each element \( g \) in a group \( G \), then \( G \) is abelian.)
8. (8 points) Let $K$ be an algebraically closed field of characteristic $p > 0$, and let $q = p^n$. Show that the solutions of the equation $x^q = x$ form a subfield $F \subseteq K$.

9. (8 points) Let $M \in \mathcal{M}_n(\mathbb{C})$ be a diagonalizable complex $n \times n$ matrix such that $M$ is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that $M$ is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

10. (8 points) Let $p > 2$ be an odd prime. Let $F$ be a field with $q = p^n$ elements. How many solutions of the equation

$$x^2 - y^2 = 1$$

are there with $x, y \in F$?

11. (8 points) Let $G$ be a group. Let $t$ be the number of subgroups of $G$ that are not normal. Prove that $t \neq 1$.

12. (9 points) Let $V$ be a vector space of dimension $n$ over a finite field $F$ with $q$ elements.

(a) Find the number of 1-dimensional subspaces of $V$.

(b) Find the number of $n \times n$ invertible matrices with entries from $F$.

(c) For each $k$, $1 \leq k \leq n$, find the number of $k$-dimensional subspaces of $V$.  


Tier 1 Examination - Algebra
January, 2014

Justify all answers!-except in problem 1. All rings are assumed to have an identity element. The set of real numbers is denoted by \( \mathbb{R} \), the complexes by \( \mathbb{C} \), and the rationals by \( \mathbb{Q} \). The symmetric group is denoted \( S_n \) and the cyclic group of order \( n \) is denoted \( C_n \).

(20)1. Find an example of each of the following (no proof necessary):
   (a) A unique factorization domain that is not a principal ideal domain.
   (b) A field with exactly 4 subfields, including the field itself.
   (c) A subgroup of \( S_4 \) that is isomorphic to \( D_4 \), the dihedral group of order 8.
   (d) A noncommutative ring \( R \) in which the only (two-sided) ideals are \{0\} and \( R \).

(10)2. Let \( G \) be a group of order \( n \). Prove that \( G \) is isomorphic to a subgroup of \( GL_n(\mathbb{C}) \).

(10)3. Let \( G \) be a finite group and let \( p \) be the smallest prime dividing \( |G| \). Prove that if \( H \) is a normal subgroup of \( G \) of order \( p \), then \( H \) is contained in the center of \( G \).

(10)4. Find, up to similarity, all the complex \( 4 \times 4 \) matrices that satisfy the polynomial \((x - 2)^2(x - 3)\) but do not satisfy \((x - 2)^3\).

(10)5. Find two elements \( A, B \) in \( GL_2(\mathbb{C}) \) each of which has finite order but such that \( AB \) does not have finite order.

(10)6. Prove that every \( n \times n \) complex matrix is similar to an upper triangular matrix.
(You may not use Jordan canonical form.)

(10)7. (a) Find the number of subgroups of \( C_5 \times C_5 \), including \( \{e\} \) and the whole group.
   (b) Find the order of the centralizer of the element \((1, 2, 3)(4, 5, 6)\) in \( S_6 \). (Recall that the centralizer of an element \( g \) in a group \( G \) is \( \{x \in G | xg = gx \} \).)

(10)8. Let \( R \) be a commutative ring in which every nonzero ideal \( I, I \neq R \), is maximal.
   (a) Prove that \( R \) has at most 2 maximal ideals.
(b) Prove if $R$ has exactly 2 maximal ideals, then there are fields $F_1, F_2$, such that $R$ is isomorphic to $F_1 \oplus F_2$.

(10)9. Let $K$ be the finite field extension of $\mathbb{Q}$ obtained by adjoining a root of the polynomial $x^6 + 3$.
(a) Prove that $K$ contains a primitive 6-th root of unity.
(b) Prove that the polynomial $x^6 + 3$ factors into linear factors in $K[x]$. 
Tier 1 Algebra Exam
August 2014

Be sure to justify all your answers. Remember to start each problem on a new sheet of paper. Each problem is worth 10 points.

(1) Consider the matrix

\[ M = \begin{bmatrix}
-\frac{1}{2} + i & 0 & \frac{1}{2} \\
-\frac{\sqrt{2}}{2} & 2 & \frac{\sqrt{2}}{2} \\
-\frac{1}{2} & 0 & \frac{1}{2} + i
\end{bmatrix} \]

(where \( i = \sqrt{-1} \))

(a) Calculate the eigenvalues of \( M \).

(b) Is \( M \) diagonalizable over \( \mathbb{C} \)? Prove it is or explain why it is not.

(c) Calculate the minimal polynomial of \( M \).

(2) (a) Let \( S_n \) denote the group of permutations of the set \( \{1, 2, \ldots, n\} \).

How many different subgroups of order 4 does \( S_4 \) have?
Justify your calculation. (Two subgroups are considered different if they are different as sets.)

(b) There is a homomorphism of \( S_4 \) onto \( S_3 \). (You do not need to prove that there exists such a homomorphism.) Show that there is no homomorphism of \( S_5 \) onto \( S_4 \).

(3) Let \( R \) be a commutative ring with unit and let \( a, b \in R \) be two elements which together generate the unit ideal. Show that \( a^2 \) and \( b^2 \) also generate the unit ideal together.

(4) Let \( \mathbb{F}_{p^n} \) denote the field with \( p^n \) elements and suppose that \( p^n - 1 = q_1^{a_1} \cdots q_k^{a_k} \) for distinct primes \( q_i \). Find the number of integers \( r \in \{0, 1, \ldots, p^n - 2\} \) for which the equation \( x^r = a \) has a solution for every \( a \in \mathbb{F}_{p^n} \).

(5) (a) Let \( G \) be a group and let \( H_1, H_2 \) be normal subgroups of \( G \) for which \( H_1 \cap H_2 = \{e\} \). Assume that any \( g \in G \) can be written \( g = h_1 h_2 \) with \( h_i \in H_i \). Show that \( G \) is isomorphic to the direct product \( H_1 \times H_2 \).

(b) Show by giving an example that the above conclusion can be false if you only assume that one of the \( H_i \) is normal.

(6) (a) Find the degree of the splitting field of the polynomial \( x^4 + 1 \) over \( \mathbb{Q} \).
(b) Find the degree of the splitting field of the polynomial $x^3 - 7$ over $\mathbb{Q}$.

(7) Show that a ring homomorphism $\phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ is an isomorphism if and only if $\phi(x) = ax + b$ for some $a, b \in \mathbb{Q}$, $a \neq 0$.

(8) For $V$ a vector space, $L : V \to V$ a linear map from $V$ to itself, and a positive integer $n$, let $L^n = L \circ L \circ \cdots \circ L$ ($n$ times).

(a) Give an example of a pair $V, L : V \to V$ so that $L$ is not the zero map, $L \neq Id$ and $L^2 = L$.

(b) Give an example of a pair $V, L$ so that $L \neq Id$ and $L^3 = Id$.

(c) Prove that if $V$ has finite dimension, then there exists an $N$ so that $\ker L^n = \ker L^N$ for all $n \geq N$.

(9) Consider the rings $\mathbb{F}_5[x]/(x^2), \mathbb{F}_5[x]/(x^2 - 3)$, and $\mathbb{F}_5 \times \mathbb{F}_5$. Show that no two of them are isomorphic to each other.

(10) (a) Show that any ring automorphism of $\mathbb{R}$ sends every element of $\mathbb{Q}$ to itself.

(b) Show that any ring automorphism of $\mathbb{R}$ sends positive numbers to positive numbers.

(c) Deduce that $\mathbb{R}$ has no nontrivial automorphisms.