Unitarity, Analyticity and Crossing Symmetry in Two- and Three-Hadron Final State Interactions

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N.B. Fuller write-up available on Workshop Resources
Motivation

“Isobar Model”

|\[A_1 t_1 + A_2 t_2|^2\]  \(t_i\) two-hadron amplitudes

Interference term \(\text{Re}[(A_1 A_2^*)(t_1 t_2^*)]\)

Extract phase of \(A_1 A_2^*\) IF know phase of \(t_1 t_2^*\)

But rescattering alters phases

Similarly for extraction of weak CP-violating phases

Three-body problem!
“Minimum Theory of FSI”

Lecture 1: Two-hadron fsi

Unitarity $\rightarrow$ $K$-matrix, $P$-vector

+ Analyticity $\rightarrow$ M-O solution, dispersion relations

Lectures 2, 3, 4: Three-hadron fsi

2: Kinematics, Dalitz plot, isobar model;
sub-energy unitarity; violated by isobar model;
U+analyticity $\rightarrow$ K-T type equations for isobar correction factors

3: Single variable integral equation for correction factors; RPE process; examples ($\pi\pi\pi$, $\pi\pi N$)

4: 3-body unitarity; particle-resonance scattering
1. Two-hadron fsi

1.1 Elastic $2 \rightarrow 2$ unitarity, $s$-waves

\[ T(s) - T^*(s) = 2i\rho(s)T^*(s)T(s) \]

\[ \rho(s) = \frac{1}{16\pi} \left( \frac{s - 4}{s} \right)^{1/2} \]

\[ T(s) = e^{i\delta} \frac{\sin \delta}{\rho} = (\rho \cot \delta - i\rho)^{-1} \]

BW: \( \rho \cot \delta = \frac{(s_R - s)}{g^2} \)

\[ T_{\text{res}} = \frac{g^2}{(s_R - s - i\rho(s)g^2)} \]

will allow $s$ to be a complex variable

\[ \arg(s - 4) \approx \pi \]

\[ s = 0 \]

\[ s = 4 \]

\[ \arg(s - 4) \approx 0 \]

\[ \arg(s - 4) \approx 2\pi \]
Physical limit for $s \geq 4$ is $s + i\epsilon$ ($s_+$)
Hermitian analyticity: $f^*(s) = f(s^*)$

\[
\text{unitarity } T_+ - T_- = 2i\rho T_+ T_-
\]

unitarity tells us the discontinuity of $T$ across the $s \geq 4$ cut

\[
U: T_+^{-1} - T_-^{-1} = -2i\rho
\]

Since $\rho_+ - \rho_- = 2\rho$, can satisfy $U$ by

\[
T^{-1} = K^{-1} - i\rho
\]

$K^{-1} \equiv \rho \cot \delta$; e.g. $K_{\text{res}} = g^2 / (s_R - s)$

\[
T = K (1 - i\rho K)^{-1} = (1 - iK\rho)^{-1}K
\]

generalizes to matrices in space of channels

Plus can add resonances in $K$, add background ..... still $T$ is unitary!

\[
e.g. \ K_{ij} = \sum_a g_{ia} g_{aj} / (s_a - s) + B_{ij}
\]
1.2 **Unitarity in 2-hadron f.s.i.**

\[
F_+ - F_- = 2i T_+ \rho F_- = 2i T_- \rho F_+ \\
\hline + \quad - \\
\hline = 2i \\
\hline + \quad - \\
\hline
\]

\[
\text{Im} F = T \rho F^* = T^* \rho F
\]

So unitarity $\rightarrow F$ must have the phase of $T$
(Watson’s Thm)
provided only one strong f.s.i.
\[ F_+ - F_- = 2iT_- \rho F_+ \Rightarrow (1 - 2iT_- \rho)F_+ = F_- \]

But \( T_- = (1 + iK\rho)^{-1}K \). So substituting,

\[
\frac{1}{1 + iK\rho}(1 - iK\rho)F_+ = F_-
\]

which by inspection is satisfied by

\[
F_+ = \frac{1}{1 - iK\rho}P
\]

where \( P \) has no branch point at \( s = 4 \).

Generalizes again to many channels.

\( P \) is a vector (which f.s. channel).

\[ e.g. \ P_i = \sum_a g_{ia} s_{a-s} f_{ap} + B_i. \]

Note \( P \) also has the resonance poles

IJRA Nucl. Phys. A 189 417 (1972)
$K_1(1270), K_2(1400) \quad P = \text{two resonances} + \text{Deck}$


\[ \alpha = \left| \frac{F_{\rho K}}{F_{K^*\pi}} \right| \]

See lectures by Ed Berger!
1.3 **Unitarity** + **analyticity**

1.3.1 Elastic $2 \rightarrow 2$ reactions

If $f(s)$ has ONLY the $s = 4$ branch point,

$$f(s) = \frac{1}{2\pi i} \int_{4}^{\infty} \frac{f(s') - f(s')}{s' - s} ds'$$

**disc** $f$ i.e. **unitarity** + **analyticity** will determine $f$ (assuming convergence)
Indeed, \( \text{disc} T^{-1} = -2i \rho \). So maybe

\[
T^{-1}(s) \equiv -\frac{1}{\pi} \int_{4\pi}^{\infty} \frac{1}{16\pi} \sqrt{\frac{s^{'2} - 4}{s'}} \frac{ds'}{s^{'2} - s} = I(s)
\]

But the integral diverges. So add one parameter, value of \( T^{-1}(s) \) at some point \( s_0 \). Then \( T^{-1}(s) = I(s_0) + [I(s) - I(s_0)] \) and the “subtracted” integral converges. Convenient to take \( s_0 = 4 \). Then

\[
T^{-1}(s) = \text{constant} + L(s)
\]

where

\[
L(s) = \frac{1}{16\pi^2} \sqrt{\frac{s-4}{s}} \ln \left( \frac{\sqrt{s-4} + \sqrt{s}}{-\sqrt{s-4} + \sqrt{s}} \right)
\]

and \( \text{Im} \ln = -\pi \) for \( s \) real, \( > 4 \)

(Chew-Madelstam function) Could still satisfy \( U \) if replace “constant” by function with no RH cut e.g. \( K^{-1} \). \( K \)-matrix with C-M phase space.
Actually $T(s)$ has “LH” cut $s \leq s_L$ as well

$$T(s) = \frac{e^{i\delta} \sin \delta}{\rho} = \frac{N(s)(L)}{D(s)(R)}$$

$$D_+ - D_- = N(T_+^{-1} - T_-^{-1}) = -2i\rho N$$

$$= -2iD_+e^{i\delta} \sin \delta$$

$$\Rightarrow D_+ = D_-e^{-2i\delta}$$

Take log of both sides

$$D(s_+) = \exp\left\{-\int_{s'}^{\infty} \frac{\delta(s')}{s' - s_+} \, ds'\right\}$$

N.B. $\frac{1}{s' - s - i\epsilon} = \frac{P.V.}{s' - s} + i\pi \delta(s' - s)$
1.3.2 Two-hadron fsi

\[ D_+ = D_- e^{-2i\delta} \]

\[ F_+ = (1 + 2i T_+ \rho) F_- = e^{2i\delta} F_- \]

Hence \( F_+ D_+ = F_- D_- \Rightarrow F(s) = C(s)/D(s) \)

where \( C(s) \) is regular at \( s = 4 \).

Now suppose we want to include “background” term (e.g. Deck) with L cut

In this case, can write

\[ F(s) = B(s) + \frac{1}{\pi} \int_4^{\infty} \frac{T^*(s') \rho(s') F(s')}{s' - s} \, ds' \]

which is an integral equation for \( F \).

Remarkably, exact solution exists (Muskhelishvili-Omnès)
M-O solution:

\[ \text{disc}_R(DF - DB) = D_+ F_+ - D_- F_- - (D_+ - D_-)B = -(D_+ - D_-)B = 2i \rho NB. \]

Hence

\[ D_+(F_+ - B) = \frac{1}{\pi} \int_4^\infty \frac{\rho(s')N(s')B(s')}{s'-s-i\epsilon} \, ds' \]

or

\[
F(s) = B(s) + \frac{1}{\pi D(s)} \int_4^\infty \frac{\rho'(s')N'(s')B'(s')}{s'-s-i\epsilon} \, ds' + C(s)/D(s)
\]

Muskhelishvili-Omnès solution
M.G. Bowler et al. N.P. B97, 227 (1975); Deck + direct + rescattering for $a_1(1260)$