Geometrical Methods for Data Analysis I: Dalitz Plots and Their Uses

- History of the Dalitz Plot
- Dalitz’s original “plot”
  non-relativistic; in terms of kinetic energies
  applied to the “\(\tau-\theta\) puzzle”
- Modern-day Dalitz Plot
- Identification of resonances
  spectroscopy -- masses, spins
- Interference Effects in Dalitz Plots

Tim Londergan
Dept of Physics and CEEM, Indiana University

Int’l Summer School on Reaction Theory
Indiana University June 8, 2015

Supported by NSF PHY-1205019
Thanks to Vincent Mathieu
The Dalitz Plot: Origin and Uses

Named after Richard Dalitz (1925-2006), professor at Chicago and Oxford
( and my thesis advisor)
Phil. Mag. 44, 1058 (1953)

Visual representation of phase space decay of a particle to various final particles -- initially only 3-body decays of spin-0 particles, but often now refers to more general decay modes

• Dalitz used it to study the “τ-θ puzzle”

• strange particles that decay to 2 or 3 pions; now understood as different decay modes of kaons

• Dalitz: “I visualize geometry better than numbers”
**Richard Dalitz**

Contributed to a wide range of scattering phenomena, spectroscopy, elementary particle physics


- “Dalitz pairs” – [electron-positron pair from neutral pion decay]
- The Dalitz Plot
- physics of Hypernuclei
- Constituent quark model
  - baryon & meson spectra
- “CDD poles” [Castillejo-Dalitz-Dyson]
Types of Reactions/Facilities

Nuclear Reactions involving exclusive final states;
Few-body final states (3, 4, … particles)

Electron machines (Jlab, BESIII, …); virtual or real photons interacting
with nucleons/nuclei

Future electron-ion collider (eRHIC; eLIC)

Electron-positron colliders; particularly those that look at exclusive
processes (e.g., BELLE, BaBar in B-quark sector)

Medium-energy antiproton accelerators (LEAR, FAIR, …)

Useful tools for spectroscopy, reaction mechanisms
Original Dalitz Plot

For three-body decay $M \to 1 + 2 + 3$, have three 4-vectors for momenta of final-state particles, and 10 constraints

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Degree of freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 four-vectors</td>
<td>12</td>
</tr>
<tr>
<td>4-momentum conservation</td>
<td>-4</td>
</tr>
<tr>
<td>3 masses</td>
<td>-3</td>
</tr>
<tr>
<td>3 Euler angles</td>
<td>-3</td>
</tr>
<tr>
<td>TOT</td>
<td>2</td>
</tr>
</tbody>
</table>

2 independent quantities! Choose kinetic energies of 2 different final particles.

Constraint: $T_1 + T_2 + T_3 = Q$
$Q = \text{energy release in decay,}$
$Q = M - m_1 - m_2 - m_3$

Choose an independent set of $T_i/Q$

(initially, in non-relativistic regime)
For three-body decay $M \rightarrow 1 + 2 + 3$, Dalitz plot used non-rel KE to plot events.
NR, use $T_i/Q$: points lie within boundary of circle of unit radius:
rel: boundary slightly distorted

\[
x = \frac{\sqrt{3}(T_1 - T_2)}{Q}; \quad y = \frac{2T_3 - T_1 - T_2}{Q}
\]
\[
Q = T_1 + T_2 + T_3 = M - m_1 - m_2 - m_3
\]
Interpreting the Dalitz Plot

Three-body decay $M \rightarrow 1 + 2 + 3$, given by square of invariant amplitude.

If invariant amplitude is constant, then Dalitz plot will be uniformly populated.

Non-uniformity in population of Dalitz plot gives information on final-state interactions in decay.

In particular, **2-body resonances** show up dramatically in the Dalitz plot.

Note: we assume all final particles are spinless here (e.g, $\pi$, $K$, $\eta$).
The Dalitz Plot and the “τ-θ Puzzle”

Observed 2 mesons, $\tau^+$ and $\theta^+$, with identical masses

$\tau^+$ had 3-body decays, $\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$
$\theta^+$ had 2-body decays, $\theta^+ \rightarrow \pi^+ + \pi^0$

**Assuming parity conservation**, spin-parity of $\theta$ would be $0^+$, $1^-$, $2^+$, $3^-$, …

Dalitz: utilize the 3-pion distributions on the Dalitz plot to determine the spin-parity of the $\tau$

Decay of $\tau$: two $\pi^+$ with angular momentum $l$, $\pi^-$ with $L$, coupled to overall $J$.

Calculate in Gottfried-Jackson frame (cm of particles 1,2); 1 and 2 have (equal + opposite) momentum $q$, particle 3 has mom. $p$
Three-pion decay of $\tau^+$ given by square of matrix element

$$\rho(x, y) = \frac{1}{2J + 1} \sum_{m_J} \left| A(m_J) \right|^2$$

$$A(m_J) = \sum_{L, \ell} [f_{L, \ell}(p, q) \otimes Y_L]^J_{m_J}$$

Assume that $f_{L,1}$ is slowly varying except for centrifugal barrier,

$$f_{L, \ell} \sim (pr)^L (qr)^\ell$$

And that for low energies only the lowest value of $L + 1$ contributes

Then we can determine the spin-parity of the $\tau$ from the distribution of points in the Dalitz plot.
Spin-Parity of the $\tau$ Particle

Form of the distribution vs. spin-parity of the $\tau$

$J^P$ dist’n

0$^-$ 1
1$^+$ $p^2$
1$^-$ $(p^2 q^2 \sin \theta \cos \theta)^2$
2$^+$ $(p q^2 \sin \theta)^2$

Dalitz plot of decays showed that $\tau$ was a 0$^-$ particle.

“$\tau$-\(\theta\) Puzzle”: “Why do 2 particles with identical masses appear to have different spin-parities?” (Dalitz, PR94, 1046 (1954))

Answer: analysis assumed $P$ conservation; once $P$ non-conservation observed (1957), these were different decays of the same particle (the $K^+$, $J^P = 0^-$)
Modern Dalitz Plot (Relativistic Kin.)

For three-body decay $M \rightarrow a + b + c$, have three 4-vectors for momenta of final-state particles, and 10 constraints

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Degree of freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 four-vectors</td>
<td>12</td>
</tr>
<tr>
<td>4-momentum conservation</td>
<td>-4</td>
</tr>
<tr>
<td>3 masses</td>
<td>-3</td>
</tr>
<tr>
<td>3 Euler angles</td>
<td>-3</td>
</tr>
<tr>
<td>TOT</td>
<td>2</td>
</tr>
</tbody>
</table>

2 independent quantities: Choose “invariant mass squared” of two different pairs.

$$x = m_{ab}^2 = s_{ab} = (p_a^\mu + p_b^\mu)^2$$

$$y = m_{ac}^2 = s_{ac} = (p_a^\mu + p_c^\mu)^2$$
Modern Dalitz Plot Kinematics

For three-body decay $M \rightarrow a + b + c$, use “invariant mass squared”

$$x = m^2_{ab} = s_{ab} = (p^\mu_a + p^\mu_b)^2$$

$$y = m^2_{ac} = s_{ac} = (p^\mu_a + p^\mu_c)^2$$

Boundaries of Dalitz Plot: $M \rightarrow 1 + 2 + 3$, look at kinematic conditions that determine boundaries: consider selected points.

Work in cm of 3-body system
Max value of $m_{13}$ corresponds to $p_2 = 0$
Moving along line of constant $m_{13}$ corresponds to constant $p_2$
Dalitz Plot, Relativistic Kinematics

\[ s_{12} = (p_1^\mu + p_2^\mu)^2 = p_1^2 + p_2^2 + p_1 \cdot p_2 \]
\[ = m_1^2 + m_2^2 + E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \]

Example: kinematics in CM of 3-particle system
For given \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \), \( (s_{12})_{\text{max}} \) occurs when \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are in opposite directions

Max value of \( s_{13} \) (top of DP) corresponds to \( \mathbf{p}_2 = 0 \)
Max value of \( \mathbf{p}_2 \) occurs at bottom of DP

If you move along a line of constant \( m_{13} \), then \( \mathbf{p}_2 \) is constant.
Similarly along lines of constant \( m_{12}, m_{23} \)
Dalitz Plot, Relativistic Kinematics

\[ s_{12} = (p_1^\mu + p_2^\mu)^2 = p_1^2 + p_2^2 + p_1 \cdot p_2 \]
\[ = m_1^2 + m_2^2 + E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2 \]

Example: kinematics in CM of 3-particle system
For given \( p_1 \) and \( p_2 \), \((s_{12})_{\text{max}}\) occurs when \( p_1 \) and \( p_2 \) are in opposite directions.

Max value of \( p_3 \) occurs on LH boundary of DP; here all 3 momenta are collinear, with 1 and 2 moving in same direction.

If you move along a line of constant \( m_{13} \), then \( p_2 \) is constant.

Q: what is kinematics at center of Dalitz Plot?
Shape of Dalitz Plot Boundary

- $Q \rightarrow 0$ (Non-relativistic regime) $\iff$ DP shape $\rightarrow$ “Egg”
- $Q \rightarrow \infty$ (Relativistic regime) $\iff$ DP shape $\rightarrow$ Triangle

$Q = \text{energy release in 3-body decay} = M - m_1 - m_2 - m_3$
Shape of Dalitz Plot Boundary

- $Q \to 0$ (Non-relativistic regime) $\Rightarrow$ DP shape $\to$ “Egg”
- $Q \to \infty$ (Relativistic regime) $\Rightarrow$ DP shape $\to$ Triangle

$m_{13}^2$ vs. $m_{12}^2$

$B^+ \to \pi^+ \pi^- \pi^+$

$J/\psi \to \pi^+ \pi^- \pi^0$

$D^+ \to \pi^+ \pi^- \pi^+$
Symmetries of Dalitz Plots

Often final-state particles are identical, in which case Dalitz plots will respect exchange symmetries

\[ \bar{p} + p \rightarrow 3 \pi^0 \]

In this reaction, since all final particles are identical bosons, DP is symmetric with respect to reflection about any of 3 axes

Three-Body Decay Through Intermediate Resonance

Three-body decays will often take place through an intermediate resonant state “$r$” that subsequently undergoes two-body decay,

$$M \rightarrow c \text{ (decays)} \quad r \rightarrow b \text{ (decays)} \quad a$$

We can describe the behavior of the resonance by using a relativistic Breit-Wigner amplitude

$$A_{BW} \sim \frac{1}{M_r^2 - s_{ab} - i\Gamma M_r} ; \quad \Gamma = \frac{\hbar}{\tau}$$

$\Gamma$ is inverse of lifetime $\tau$ of resonant state
Isobar Model for 3-Body Decays

Approximate the total 3-body decay amplitude as a coherent sum of processes where one particle is a spectator (plus background?)

\[ A_D(s_{12}, s_{13}) = a_0 e^{i\delta_0} + \sum_r a_r e^{i\delta_r} A_r(s_{12}, s_{13}) \]
2-Body Resonances in Dalitz Plots

A 2-body resonance will appear on a Dalitz plot as a sharp enhancement, corresponding to the pair of particles that forms the resonance

$$A_{BW}(ab) \sim \frac{1}{M_r^2 - s_{ab} - i\Gamma M_r}$$
Identification of Resonances in Dalitz Plot

Use relativistic Breit-Wigner form for resonance,

$$A_{BW}(ab) \sim \frac{1}{M_r^2 - s_{ab} - i\Gamma M_r}$$

$\Gamma$ is inverse of lifetime of resonant state

![Magnitude and Phase Plots](image-url)
Narrow and Broad Resonances in Dalitz Plot

Magnitude

\[ A_{BW}(ab) \sim \frac{1}{M_r^2 - s_{ab} - i\Gamma M_r} \]
Resonance Spin in Dalitz Plot

If resonant state has spin $S$ and $a$, $b$ and $c$ are all spinless particles, then decay amplitude is proportional to Legendre Polynomial;

In Gottfried-Jackson frame (rest frame of $[ab]$ pair),

$$A \sim A_{BW} P_S(\cos \theta) ;$$

$$P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1)$$
Resonance Spin in Dalitz Plot

If resonance has spin $S$ and $a$, $b$ and $c$ are all spinless particles, then decay amplitude will have zeroes corresponding to Legendre Polynomial;

$$A \sim A_{BW} P_S(\cos \theta) ;$$

$$P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1)$$

Spin-S resonance will have $S$ zeroes in Dalitz plot
Interference of Overlapping Resonances in Dalitz Plot

Constructive interference between a pair of resonances in \([ab]\)
Interference of Resonances in Dalitz Plot (2)

Destructive interference between a pair of resonances in \([ab]\)
Applications of Isobar Model

Total 3-body decay amplitude = coherent sum of 2-body resonant processes plus background?

\[ \mathcal{A}_D(s_{12}, s_{13}) = a_0 e^{i\delta_0} + \sum_r a_r e^{i\delta_r} \mathcal{A}_r(s_{12}, s_{13}) \]

Relativistic Breit-Wigner parameters extracted from resonance properties (or from particle data tables);
\( a_i, \delta_i \) are constants that are determined through a maximum-likelihood fit
Can measure fractions and relative phases of different isobars
A Current Puzzle

Compare decays $J/\psi \rightarrow \pi^+ + \pi^- + \pi^0$ to $\psi' \rightarrow \pi^+ + \pi^- + \pi^0$

The "$\rho$-\pi puzzle"

$J/\psi, \psi'$ both $c\bar{c}$ bound states, but $3\pi$ decays very different

$J/\psi$: almost exclusively through $\rho (770)$

$\psi'$ decays through cluster of states $\sim 2.2$ GeV

Higher-Order Dalitz Plots

Three-body decay $M \rightarrow 1 + 2 + 3$, with 2 scalar and 1 vector final particles.

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Degree of freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 four-vectors</td>
<td>12</td>
</tr>
<tr>
<td>4-momentum conservation</td>
<td>-4</td>
</tr>
<tr>
<td>3 masses</td>
<td>-3</td>
</tr>
<tr>
<td>3 Euler angles</td>
<td>-3</td>
</tr>
<tr>
<td>Vector helicity</td>
<td>2</td>
</tr>
<tr>
<td><strong>TOT</strong></td>
<td><strong>4</strong></td>
</tr>
</tbody>
</table>

Now 4 independent quantities! 4-D phase space, “4-D Dalitz Plot”

Recently used because decays to vector + scalars showed enhancements.

Must use helicity formalism

$B^0 \rightarrow \psi(2S) K \pi$  $B^0 \rightarrow \chi_{c1} K \pi^+$
4-D Dalitz Plots

Three-body decay $M \rightarrow 1 + 2 + 3$, with 2 scalar and 1 vector final particles.

Shadows? Interference? Or, new states?
Detection and Spin of Resonances in Dalitz Plot

Here is a Dalitz plot of a 3-body decay
• How many resonances can you detect?
• What is the spin of each resonance?
• Which states have higher masses?
Detection and Spin of Resonances in Dalitz Plot

Decay \( D^0 \rightarrow K_s \pi^+ \pi^- \)

Because both x and y axes have symmetric about reflection (diagonal = \( \pi + \pi \))

- green + blue: \( K^* (892), S=1 \)
- cyan + magenta: \( K_2^* (1430), S=2 \)
- yellow: \( \rho (770), S=1 \)
- red: \( f_0 (980), S=0 \)
Conclusions:

Dalitz Plots convert 3-body decays into plots that allow one to intuit important processes driving decays:

- Any non-random processes are highlighted in DP
- Resonances occur as **sharp bands in DP**
  - can read off **position, width** of resonance
  - **spin of resonance** determined by zeroes in DP
- Can also determine branching fractions, phases
- High-energy physics: now measuring CP-violating phases, etc. using Dalitz plots

Dalitz Plot: one of the most useful tools in particle spectroscopy