Dispersion relations: some applications
Light quark masses from $\eta \to 3\pi$

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1.3 QCD at low energy

- At low energy, impossible to describe QCD with perturbation theory since $\alpha_s$ becomes large.

Need non-perturbative methods.

- Two model independent methods:
  - Effective field theory
    Ex: ChPT for light quarks
  - Numerical simulations on the lattice

[Bethke, EPJC’09]
1.4 Chiral Symmetry

- Limit $m_k \to 0$

\[
\mathcal{L}_{QCD} \to \mathcal{L}_{QCD}^0 = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{q}_L i \gamma^\mu D_\mu q_L + \bar{q}_R i \gamma^\mu D_\mu q_R, \quad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}
\]

with $q_{L/R} \equiv \frac{1}{2} (1 \pm \gamma_5)q$

Symmetry: $G \equiv SU(3)_L \otimes SU(3)_R \to SU(3)_V$

- $G$ spontaneously broken, ground state not invariant under $G \equiv SU(3)_L \otimes SU(3)_R$ but invariant under $SU(3)_{V=L+R}$

**Goldstone bosons** with quantum numbers of pseudoscalar mesons are generated

$\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$ massless states
1.5 Construction of an effective theory: ChPT

- Degrees of freedom: **Goldstone bosons** (GB)
  - Symmetry group: $G \equiv SU(3)_L \otimes SU(3)_R$

- Build all the corresponding invariant operators including explicit symmetry breaking parameters
  - $\mathcal{L}_{ChPT} \equiv \mathcal{L}(U, \chi)$
  - GB’s Masses $\sim m_q$

- Goldstone bosons interact weakly at low energy and $m_u, m_d \ll m_s < \Lambda_{QCD}$
  - Expansion organized in external momenta and quark masses
  - Weinberg’s power counting rule

$$\mathcal{L}_{\text{eff}} = \sum_{d \geq 2} \mathcal{L}_d, \mathcal{L}_d = \mathcal{O}(p^d), p \equiv \{q, m_q\}$$

$$p \ll \Lambda_H = 4\pi F_\pi \sim 1 \text{ GeV}$$
1.6 Chiral expansion

- \( \mathcal{L}_{\text{ChPT}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \)
  - \( \text{LO : } \mathcal{O}(p^2) \)
  - \( \text{NLO : } \mathcal{O}(p^4) \)
  - \( \text{NNLO : } \mathcal{O}(p^6) \)

- Renormalizable and unitary order by order in the expansion

- The structure of the lagrangian is fixed by chiral symmetry but not the coupling constants \( \rightarrow \) LECs appearing at each order

  \[ \mathcal{L}_2 : F_0, B_0, \quad \mathcal{L}_4 = \sum_{i=1}^{10} L_i O_4^i, \quad \mathcal{L}_6 = \sum_{i=1}^{90} C_i O_6^i \]

- LECs describe the influence of heavy degrees of freedom not contained in the ChPT lagrangian

- Naturalness: LECs of order one
1.6 Chiral expansion

- The LECs calculable if QCD solvable, instead
  - Determined from experimental measurement
  - Estimated with models: Resonances, large $N_C$
  - Computed on the lattice

- In a specific process, only a limited number of LECs appear
1.6 Chiral expansion

- Ex: $\eta \rightarrow \pi^+ \pi^- \pi^0 \Rightarrow A = A_2 + A_4 + A_6 + \ldots$

- Tree level $\mathcal{O}(p^2)$:

- One loop $\mathcal{O}(p^4)$:

$\pi\pi \rightarrow \pi\pi$ at tree
1.6 Chiral expansion

- Ex: $\eta \rightarrow \pi^+ \pi^- \pi^0 \Rightarrow A = A_2 + A_4 + A_6 + \ldots$

- Tree level $\mathcal{O}(p^2)$:

- One loop $\mathcal{O}(p^4)$:

$\pi \pi \rightarrow \pi \pi$ at tree
Comparison of values of $Q$ from Dashen corrections

![Graph showing comparison of $Q$ values from different authors]

1. The strong interaction as a quantum field theory
2. Dashen, Duncan et al., Bijnens & Prades, Donoghue & Perez
3. Ananthanarayan & Moussallam

Figure 1.4: $Q$ as calculated from a meson ratio with different values for the electromagnetic kaon mass splitting. The left-most point has been calculated in the absence of Dashen violation and thus agrees with $Q_{Dashen}$. The other points, from left to right, have been taken from Refs. [68–71]. The figure has been inspired by Kaplan and Manohar [73].

This is the value of $Q$ in the absence of Dashen violation. The electromagnetic kaon mass splitting $\Delta m^\text{em}_K$ is substantially changed by higher order effects. Several authors have calculated Dashen violating contributions, e.g., Refs. [68–71], and found deviations from Dashen's theorem that range from 50 up to 150 per cent. Figure 1.4 shows their values for $\Delta m^\text{em}_K$ together with the corresponding results for $Q$.

Kaplan and Manohar [73] have shown that changes in the quark masses of the form $m_u \rightarrow m_u + \alpha m_d m_s$ (and cyclic) can be absorbed into $O(p^4)$ operators by shifting the low-energy constants $L_6$, $L_7$, and $L_8$ accordingly. The quark mass ratios $m_u/m_d$, $m_s/m_d$, and $R$ are not invariant under this transformation which implies that corrections from $L_4$ can, in principle, change them to any value that can be reached by the aforementioned shift of the quark masses. The double ratio $Q$, on the other hand, is not affected by the transformation up to corrections of $O(M^2)$. The transformation of the quark masses depends on a single parameter $\alpha$, such that the ratios $m_u/m_d$ and $m_s/m_d$ are not independent. They are rather constrained to lie on an
Comparison of values of $Q$

$\eta \rightarrow \pi^+ \pi^- \pi^0$ decays
- Anisovitch & Leutwyler’96
- Kambor, Wiesendanger & Wyler’95

Kaon mass splitting
- no Dashen violation
  - Weinberg’77
- Lattice
  - Ducan et al.’96
- ENJL model
  - Bijnens & Prades’97
- VMD
  - Donoghue & Perez’97
- Sum rules
  - Anant & Moussallam’04
- Sum rules, update
  - Kastner & Neufeld’08

$Q = 22.3 \pm 0.9$
$Q = 22.1 \pm 0.9$
$Q = \pm 22.3 \pm 0.9$
$Q = \pm 22.1 \pm 0.9$

Fair agreement with the determination from meson masses
\[ I^G(J^{PC}) = 0^+(0^--) \]

Mass \( m = 547.862 \pm 0.018 \) MeV
Full width \( \Gamma = 1.31 \pm 0.05 \) keV

**C-nonconserving decay parameters**
- \( \pi^+ \pi^- \pi^0 \) left-right asymmetry = \((0.09^{+0.11}_{-0.12}) \times 10^{-2}\)
- \( \pi^+ \pi^- \pi^0 \) sextant asymmetry = \((0.12^{+0.10}_{-0.11}) \times 10^{-2}\)
- \( \pi^+ \pi^- \pi^0 \) quadrant asymmetry = \((-0.09 \pm 0.09) \times 10^{-2}\)
- \( \pi^+ \pi^- \gamma \) left-right asymmetry = \((0.9 \pm 0.4) \times 10^{-2}\)
- \( \pi^+ \pi^- \gamma \)\( \beta \) (D-wave) = \(-0.02 \pm 0.07 \) (\( S = 1.3 \))

**CP-nonconserving decay parameters**
- \( \pi^+ \pi^- e^+ e^- \) decay-plane asymmetry \( A_\phi = (-0.6 \pm 3.1) \times 10^{-2} \)

**Dalitz plot parameter**
- \( \pi^0 \pi^0 \pi^0 \) \( \alpha = -0.0315 \pm 0.0015 \)
<table>
<thead>
<tr>
<th>Decay Modes</th>
<th>Fraction ($\Gamma_j/\Gamma$)</th>
<th>Scale factor/Confidence level</th>
<th>$p$ (MeV/c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\gamma$</td>
<td>$(39.41 \pm 0.20)$ %</td>
<td>$S=1.1$</td>
<td>274</td>
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<tr>
<td>$3\pi^0$</td>
<td>$(32.68 \pm 0.23)$ %</td>
<td>$S=1.1$</td>
<td>179</td>
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<tr>
<td>$\pi^0\pi^0\gamma$</td>
<td>$(2.7 \pm 0.5) \times 10^{-4}$</td>
<td>$S=1.1$</td>
<td>257</td>
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<tr>
<td>$2\pi^0\gamma$</td>
<td>$&lt; 1.2 \times 10^{-3}$</td>
<td>$S=90%$</td>
<td>238</td>
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<tr>
<td>$4\gamma$</td>
<td>$&lt; 2.8 \times 10^{-4}$</td>
<td>$S=90%$</td>
<td>274</td>
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<td>invisible</td>
<td>$&lt; 1.0 \times 10^{-4}$</td>
<td>$S=90%$</td>
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**Charged modes**

<table>
<thead>
<tr>
<th>Charged modes</th>
<th>Fraction ($\Gamma_j/\Gamma$)</th>
<th>Scale factor/Confidence level</th>
<th>$p$ (MeV/c)</th>
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</thead>
<tbody>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$(28.10 \pm 0.34)$ %</td>
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<tr>
<td>$\pi^+\pi^-\gamma$</td>
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<tr>
<td>$e^+e^-\gamma$</td>
<td>$(4.22 \pm 0.08)$ %</td>
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<td>236</td>
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<tr>
<td>$\mu^+\mu^-\gamma$</td>
<td>$(6.9 \pm 0.4) \times 10^{-3}$</td>
<td>$S=1.3$</td>
<td>274</td>
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<tr>
<td>$e^+e^-\mu^+\mu^-$</td>
<td>$(3.1 \pm 0.4) \times 10^{-4}$</td>
<td>$S=1.3$</td>
<td>253</td>
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<tr>
<td>$\mu^+\mu^-$</td>
<td>$&lt; 5.6 \times 10^{-6}$</td>
<td>$S=90%$</td>
<td>274</td>
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<tr>
<td>$2e^+2e^-$</td>
<td>$(5.8 \pm 0.8) \times 10^{-6}$</td>
<td>$S=90%$</td>
<td>253</td>
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<tr>
<td>$\pi^+\pi^-\gamma$</td>
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<td>$S=90%$</td>
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<tr>
<td>$\pi^0\mu^+\mu^-$</td>
<td>$&lt; 3.6 \times 10^{-4}$</td>
<td>$S=90%$</td>
<td>210</td>
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**Charge conjugation ($C$), Parity ($P$), Charge conjugation $\times$ Parity ($CP$), or Lepton Family number ($LF$) violating modes**

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43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND $d$ FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

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<td>-1/2</td>
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$Y^0_1 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$Y^1_1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y^1_0 = \sqrt{\frac{5}{4\pi}} (\frac{3}{2} \cos^2 \theta - \frac{1}{2})$

$Y^2_0 = -\frac{15}{8\pi} \sin \theta \cos \theta e^{i\phi}$

$Y^2_1 = \frac{\sqrt{15}}{2\pi} \sin^2 \theta e^{2i\phi}$

$Y^2_2 = \sqrt{\frac{15}{2\pi}} (2 \cos \theta - 1)$

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle = (-1)^{j_1-j_2} (j_2 j_1 m_2 m_1 | j_2 j_1 JM \rangle$
4.2 Method: Representation of the amplitude

- **Decomposition** of the amplitude as a function of isospin states

\[ M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \]

- **Fuchs, Sazdjian & Stern’93**
- **Anisovich & Leutwyler’96**

- \( M_I \) isospin \( I \) rescattering in two particles

- Amplitude in terms of S and P waves exact up to NNLO (\( \mathcal{O}(p^6) \))

- Main two body rescattering corrections inside \( M_I \)

- Functions of only one variable with only right-hand cut of the partial wave

\[ \text{disc} \left[ M_I(s) \right] = \text{disc} \left[ f^I_{\ell}(s) \right] \]

- **Elastic unitarity** \( \text{Watson’s theorem} \)

\[ \text{disc} \left[ f^I_{\ell}(s) \right] \propto t^*_\ell(s)f^I_{\ell}(s) \]

with \( t_\ell(s) \) partial wave of elastic scattering
4.4 Dispersion Relations for the $M_I(s)$

- Elastic Unitarity

$$\text{disc}[M_I] = \text{disc}[f^I_1(s)] = \theta(s - 4M^2_\pi)\left[ M_I(s) + \hat{M}_I(s) \right] \sin \delta^I_1(s) e^{-i\delta^I_1(s)}$$

$\delta^I_1$ phase of the partial wave $f^I_1(s)$

- Watson theorem: elastic $\pi\pi\pi$ scattering phase shifts

- Solution: Inhomogeneous Omnès problem

$$M_0(s) = \Omega_0(s) \left( \alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi} \int_0^\infty ds' \frac{\sin \delta^0_0(s') \hat{M}_0(s')}{\Omega_0(s')(s' - s - i\epsilon)} \right)$$

Omnès function

Similarly for $M_1$ and $M_2$
3.3 Dispersion Relations for the $M_I(s)$

- Unitary relation for $M_I(s)$:

$$\text{disc } M_I(s) = 2i \left( M_I(s) + \right) t^*_{\pi\pi\rightarrow\pi\pi}(s) \rho(s) \theta\left(s - 4M^2_\pi\right)$$

Elastic Unitarity

- Right-hand cut only  Omnès problem

$$M_I(s) = P_I(s) \Omega_I(s)$$

$$\left[ \Omega_I(s) = \exp\left(\frac{s}{\pi} \int_0^\infty ds' \frac{\delta_I(s')}{s'(s' - s - i\epsilon)}\right) \right]$$

- Watson’s theorem in the elastic region: Inputs needed: S and P-wave phase shifts of $\pi\pi$ scattering

Emilie Passemard
Inputs: $\pi\pi$ scattering

- S wave

- P wave

$\pi\pi$ phase shifts extracted combining all experimental results solving Roy equations → A large number of theoretical analyses Ananthanarayan et al’01, Colangelo et al’01, Descotes-Genon et al’01, Garcia-Martin et al’09,’11, Colangelo et al.’11 and all agree
3.3 Dispersion Relations for the $M_I(s)$

- Unitary relation for $M_I(s)$:

$$\text{disc } M_I(s) = 2i \left( M_I(s) + \hat{M}_I(s) \right) \sin \delta_I(s) e^{-i\delta_I(s)} \theta(s - 4M^2_\pi)$$

right-hand cut

left-hand cut

- Dispersion relation for the $M_I$'s

$$M_I(s) = \Omega_I(s) \left( P_I(s) + \frac{s^n}{\pi 4M^2_\pi} \int_{4M^2_\pi}^\infty ds' \frac{\sin \delta_I(s') \hat{M}_I(s')}{\Omega_I(s')} \left( s' - s - i\varepsilon \right) \right)$$

Omnès function

- $\hat{M}_I(s)$: singularities in the t and u channels, depend on the other $M_I(s)$

Crossed-channel scattering between s-, t-, and u-channel

Angular averages of the other functions

Coupled equations

Khuri & Treiman’60
Aitchison’77
Anisovich & Leutwyler’98
Hat functions

• Subtract $M_1(s)$ from the partial wave projection of $M(s,t,u)$

\[
\hat{M}_0(s) = \frac{2}{3} \langle M_0 \rangle + 2(s-s_0)\langle M_1 \rangle + \frac{20}{9} \langle M_2 \rangle + \frac{2}{3} \kappa(s) \langle z M_1 \rangle
\]

Non trivial angular averages need to deform the integration path to avoid crossing cuts.

Generates complex analytic structure (3-particle cuts)

Anisovich & Anselm’66
1.4 Determination of the form factors: $F_\pi(s)$

- Cauchy Theorem: build the FF in the entire phase space

\[
F(s) = \frac{1}{2i\pi} \oint_C \frac{F(s')}{(s'-s)} ds' \\
= \frac{1}{\pi} \int_{s_{th}}^{\Lambda^2} ds' \frac{\text{disc}(F(s))}{s'-s-i\varepsilon} + \frac{1}{2i\pi} \oint_{s=|\Lambda^2|} ds' \frac{F(s')}{s'-s}
\]

As $\Lambda \to \infty$

\[
F(s) = \frac{1}{\pi} \int_{4M^2_\pi}^{\infty} \frac{\text{disc}[F(s')]}{s'-s-i\varepsilon} ds'
\]

Dispersion Relation
4.4 Dispersion Relations for the $M_I(s)$

- Similarly for $M_1$ and $M_2$
  - Four subtraction constants to be determined: $\alpha_0$, $\beta_0$, $\gamma_0$ and one more in $M_1 (\beta_1)$
  - Inputs needed for these and for the $\pi\pi$ phase shifts $\delta^I_\ell$
    - $M_0$: $\pi\pi$ scattering, $\ell=0$, $I=0$
    - $M_1$: $\pi\pi$ scattering, $\ell=1$, $I=1$
    - $M_2$: $\pi\pi$ scattering, $\ell=0$, $I=2$
  - Solve dispersion relations numerically by an iterative procedure
3.4 Iterative Procedure

set $M_i$ to tree-level

calculate $\hat{M}_i$ from $M_i$

calculate $M_i$ from $\hat{M}_i$

accuracy reached?

Yes

done

No

fix subtraction constants
3.5 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler’96*

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]
\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]
\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 \]

- In the work of *Anisovich & Leutwyler’96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  ➡️ The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA and MAMI
  ➡️ Use the data to directly fit the subtraction constants

- Solution *linear* in the *subtraction constants* *Anisovich & Leutwyler’96*

\[ M(s,t,u) = \alpha_0 M_{\alpha_0} (s,t,u) + \beta_0 M_{\beta_0} (s,t,u) + \ldots \]
  ➡️ makes the fit much easier
Experimental measurements

- Dalitz plot measurement: Amplitude expanded in X and Y around X=Y=0

\[
|A(s,t,u)|^2 = \Gamma(X,Y) = N \left(1 + aY + bY^2 + dX^2 + fY^3\right)
\]

\[
X = \frac{\sqrt{3} (T_+ - T_-)}{Q_c} = \frac{\sqrt{3}}{2M\eta Q_c} (u - t)
\]

\[
Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M\eta Q_c} \left(\left(M\eta - M_{\pi^0}\right)^2 - s\right) - 1
\]

with \(T_i\): kinetic energy of \(\pi^i\) in the \(\eta\) rest frame

and \(Q_c \equiv T_0 - T_+ - T_- = M\eta - 2M_{\pi^+} - M_{\pi^0}\)
Experimental measurements: Charged channel

- Charged channel measurements with high statistics from KLOE and WASA
  - e.g. **KLOE**: $\sim 1.3 \times 10^6 \eta \rightarrow \pi^+ \pi^- \pi^0$ events from $e^+e^- \rightarrow \phi \rightarrow \eta \gamma$

  \[
  \left| A_c(s,t,u) \right|^2 = N \left( 1 + aY + bY^2 + dX^2 + fY^3 \right)
  \]

\[\begin{align*}
  Y &= \frac{3}{2M_\eta Q_c} \left( \left( M_\eta - M_{\pi^0} \right)^2 - s \right) - 1 \\
  X &= \frac{\sqrt{3}}{2M_\eta Q_c} (u - t)
\end{align*}\]
Experimental measurements : Neutral channel

- Neutral channel measurements with high statistics from MAMI-B, MAMI-C and WASA e.g. MAMI-C: \( \sim 3 \times 10^6 \eta \rightarrow 3\pi^0 \) events from \( \gamma p \rightarrow \eta p \)

\[
\left| A_n(s,t,u) \right|^2 = N \left( 1 + 2\alpha Z + 6\beta Y \left( \frac{X^2 - Y^2}{3} \right) + 2\gamma Z^2 \right)
\]

Extraction of the slope:

\[
Z = \frac{2}{3} \sum_{i=1}^{3} \left( \frac{3T_i}{Q_n} - 1 \right)^2 = X^2 + Y^2
\]

\[
Q_n \equiv M_\eta - 3M_{\pi^0}
\]

\[
X = \sqrt{3} \frac{(T_+ - T_-)}{Q_c} = \sqrt{3} \frac{(u - t)}{2M_\eta Q_c}
\]

\[
Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c} \left( \left( M_\eta - M_{\pi^0} \right)^2 - s \right) - 1
\]
Experimental measurements: Neutral channel

- Neutral channel measurements with high statistics from **MAMI-B, MAMI-C** and **WASA** e.g. **WASA**: \(\sim 1.2 \times 10^5 \ \eta \rightarrow 3\pi^0\) events from \(pp \rightarrow \eta pp\)

\[
|A_n(s, t, u)|^2 = N \left(1 + 2\alpha Z + 6\beta Y \left(X^2 - \frac{Y^2}{3}\right) + 2\gamma Z^2\right)
\]

![Graph](image)

\(\alpha = -0.027 \pm 0.008\) (stat) \(\pm 0.005\) (syst)

**Cusp effect**

*Gullstrom, Kupsc, Rusetsky’09*

**WASA’09**

\[
X = \frac{3}{2M_\eta Q_c} \left(\frac{T_+ - T_-}{Q_c}\right) = \frac{\sqrt{3}}{2M_\eta Q_c} (u - t)
\]

\[
Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c} \left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1
\]
3.4 Subtraction constants

- As we have seen, only Dalitz plots are measured, *unknown normalization!*

\[ A(s,t,u) = -\frac{1}{Q^2} \frac{M_K^2}{M^2} \frac{M_K^2 - M^2}{3\sqrt{3}F^2}s^3 M(s,t,u) \]

\( Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \)

To determine Q, one needs to know the normalization

- For the normalization one needs to use ChPT

- The subtraction constants are

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]

\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]

\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 \]

Only 6 coefficients are of physical relevance
3.4 Subtraction constants

- The subtraction constants are

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]
\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]
\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 \]

Only 6 coefficients are of physical relevance

- They are determined from
  - Matching to one loop ChPT \( \Rightarrow \) \( \delta_0 = \gamma_1 = 0 \)
  - Combine ChPT with fit to the data \( \Rightarrow \) \( \delta_0 \) and \( \gamma_1 \) are determined from the data

- Matching to one loop ChPT: Taylor expand the dispersive \( M_1 \)
  - Subtraction constants \( \Leftrightarrow \) Taylor coefficients
Dispersive approach

- Dispersion Relations: extrapolate ChPT at higher energies
  - Important corrections in the physical region taken care of by the dispersive treatment!
4.3 Qualitative results of our analysis

- Plot of $Q$ versus $\alpha$:

- All the data give consistent results. The preliminary outcome for $Q$ is intermediate between the lattice result and the one of Kastner and Neufeld.

**NB:** Isospin breaking has not been accounted for.

From kaon mass splitting:

$$Q = 20.7 \pm 1.2$$

*Kastner & Neufeld’08*
Isospin violating process possibility to extract the quark mass ratio $Q$:

$$\Gamma_{\eta \to 3\pi} \propto \int |A(s,t,u)|^2 \propto Q^{-4}$$

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \left[ \hat{m} \equiv \frac{m_d + m_u}{2} \right]$$

$$A(s,t,u) = \frac{N}{Q^2} M(s,t,u)$$

- $M(s,t,u)$ determined through the dispersive analysis of the data but for $N$ one has to rely on ChPT

Analysis for JPAC by P. Guo, I. Danilkin, D. Schott et al’15 using WASA data

$Q = 21.4 \pm 0.4$ Analysis of CLAS data

G. Colangelo, S. Lanz, H. Leutwyler, E.P., in progress

dispersive (Walker)
dispersive (Kambor et al.)
dispersive (Kampf et al.)$
\chi$PT $O(p^4)$
$\chi$PT $O(p^6)$

no Dashen violation with Dashen violation

lattice (FLAG average)
dispersive, one loop
dispersive, fit to KLOE

Preliminary
2.4 Results: quark mass ratios

\[ Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \]

- lattice
- intersection
- \( \eta \) decay (preliminary)

H.Leutwyler
5. Back-up
1.6 Chiral expansion

- Ex: $\eta \rightarrow \pi^+ \pi^- \pi^0 \Rightarrow A = A_2 + A_4 + A_6 + \ldots$

- Tree level $\mathcal{O}(p^2)$:

- Two loops $\mathcal{O}(p^6)$:

- $A_2$

- $A_6$
2.2 Extraction of $Q$

- Extraction of the quark masses:

\[ \Gamma_{\eta \to 3\pi} \propto Q^{-4} |M|^2 \]

- Computed with dispersive methods and ChPT

Experiment:
- KLOE (Italy),
- MAMI (Germany),
- WASA (Sweden, Germany),
- CLAS (JLab, USA)

- Dispersive method: Take into account the $\pi\pi$ final state interactions

\[ Q^2 \propto (m_u - m_d) \]
Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3} \langle M_0 \rangle + 2(s - s_0) \langle M_1 \rangle + \frac{20}{9} \langle M_2 \rangle + \frac{2}{3} \kappa(s) \langle z M_1 \rangle$

where $\langle z^n M_I \rangle(s) = \frac{1}{2} \int_{-1}^{1} dz \ z^n M_I(t(s, z))$, $z = \cos \theta$ scattering angle

Non trivial angular averages need to deform the integration path to avoid crossing cuts

Anisovich & Anselm’66
Discontinuities of the $M_I(s)$

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where $\langle z^n M_I \rangle(s) = \frac{1}{2} \int_{-1}^{1} dz \, z^n M_I(t(s,z))$, $z = \cos \theta$ scattering angle

Non trivial angular averages need to deform the integration path to avoid crossing cuts

Anisovich & Anselm’66
3.7 Comparison of values of $Q$

Fair agreement with the determination from meson masses