Veneziano model and application to Dalitz plot analysis

Motivation

Properties

Application to $J/\psi \rightarrow 3\pi$ decays

Generalizations (dual models)
2-to-2 kinematics
e.g. $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0 : A(s,t,u)$

\[ s = (1 + 2)^2 \quad t = (1 + 3)^2 \quad u = (1 + 4)^2 \]

\[ s + t + u = 4m^2 \]

physical domain in the s-channel
\[ E_3^\pm = -E_3 > 0 \]
\[ E_4^\pm = -E_4 > 0 \]

physical domain in the t-channel
\[ E_2^\pm = -E_2 > 0 \]
\[ E_4^\pm = -E_4 > 0 \]
$$A(s, t) = \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(z_s)$$

$$z_s = 1 + \frac{2t}{s - 4m^2}$$

$$A(s, t) = \sum_{l=0}^{\infty} (2l + 1) A_l(t) P_l(z_t)$$

$$z_t = 1 + \frac{2s}{t - 4m^2}$$

to reproduce peaks in t (or s) need to continue the s (or t) channel p.w. sum outside its domain of convergence.
peaks from poles

zeros from $P_i$'s

$s = m_r^2$

large $-z_s$

large $+z_s$

$t = m_r^2$

$u = m_r^2$

$s = 1 + \frac{2t}{s - 4m^2}$
Analytical continuation: simple example

\[ A(s, t) = A(s, t(s, z_s)) = \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(z_s) \]

\[ t(s, z_s) = -(1 - z_s) \frac{s - 4m^2}{2} \]

\[ |z_s| < 1 \text{ in the s-channel and } z_s < -1 \text{ in the t-channel} \]

\[ A(z_s) = \sum_{l=0}^{\infty} (-z_s)^l = 1 - z_s + z_s^2 + \cdots \]

well defined for \(|z_s| < 1\)

\[ A(z_s) = \frac{1}{1 + z_s} \]
for large $-z_s$

$$A(z_s) = \frac{1}{1 + z_s} = (z_s)\alpha \left[1 + O(1/z_s)\right]$$

$\alpha = -1$

compare with starting point

$$A(s, t) = A(s, t(s, z_s)) = \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(z_s)$$

$$t(s, z_s) = -\left(1 - z_s\right) \frac{s - 4m^2}{2}$$

we derived the large $+t$ behavior!

- there is a pole in the physical $t$ region ("resonance")
- if $\alpha$ non-integer, tasks as a function of $t$ the amplitude has a branch point: particle production at lathe $t$-channel energy
- the leading term at large $t$ is "simple": it must come from some specific property of the p.w. series

Exercise: find the large-$z_s$ behavior using the Sommerfeld-Watson transform.
Analytical continuation: realistic example

\[ A(s, t) = \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(z_s) = \sum_{l=0}^{\infty} \frac{\beta(s)}{l - \alpha(s)} \left[ P_l(z_s) - P_l(-z_s) \right] / 2 \]

\[ \alpha(s) = \frac{1}{2} + s + i\gamma \rho(s) \quad \rho(s) \beta(s) = Im\alpha(s) \quad (*) \]

\[ \gamma \approx \Gamma \rho m_\rho \]

\[ \rho(s) = \sqrt{1 - 4m_\pi^2 / s} \]

- \( A_1(s) \sim \text{Breit-Wigner of the rho-meson} \)
- Re(\( \alpha(s) \)) = 1/2 + s : linear Regge trajectory
- Im(\( \alpha(s) \)) = is related to resonance widths
- the relation between \( \alpha \) (trajectory) and \( \beta \) (residue) follows from unitarity: Im \( A_1(s) = |A_1(s)|^2 \rho(s) \)
- Resonances with different spins in \( A_1, A_3, A_5, \ldots \) are related by poles in \( l \) of the function \( A_l \)

**Exercise**: show that

\[ \sum_{l=0}^{\infty} \frac{z_s^l \pm (-z_s)^l}{l - \alpha} \propto \frac{1 \pm e^{i\pi \alpha}}{\sin \pi \alpha} (-z_s)^{\alpha} \]

**Exercise**: show that \((*)\) follows from unitarity

**Exercise**: show that \((**)\) has a Breit-Wigner form
• the leading term at large $t$ is “simple”: it must come from some specific property of the p.w. series - it comes from right most singularity of partial waves in the angular momentum plane

$$
\sum_{l=0}^{\infty} \frac{z^l_s \pm (-z_s)^l}{l - \alpha} \propto \frac{1 \pm e^{i\pi\alpha}}{\sin \pi\alpha} (-z_s)^\alpha
$$

Regge theory = origin and properties of singularities of p.w. in the angular momentum plane

• it is possible (“simple”) to construct models of partial waves in one channel (e.g. $s$) which have Regge poles and produce right asymptotic behavior in another (e.g. $t$). It is not easy to do it simultaneously
**s-channel p.w. : s-channel resonances**

\[ A(s,t) = \sum_{l}^{\infty} (2l + 1) A_{l}(s) P_{l}(s), \quad A_{l}(s) = \sum_{i=\text{Regge poles}} \frac{\beta_{i}(s)}{l - \alpha_{i}(s)} \]

**Q: Should you add t-channel resonances**

(interference model) ?

**A: No. (resonances in t and s are dual not additive)**

A finite number of  t-channel resonances will break s-channel analyticity

\[ \sum_{l=0}^{L_{\text{max}}} (2l + 1) A_{l}(t) P_{l}(z_{t}) \rightarrow \text{s}^{L_{\text{max}}} \]

An infinite number of t-channel resonance -> double counting of A(s,t)

**Veneziano model =** has simultaneous resonances in s and channel and proper asymptotic behavior. To do this requiring number of p.w/resonances (why?)
What functions have an infinite number of poles (resonances) to be used to represent (model) the amplitude $A(s,t)$?

The Gamma function !!!
\[ \Gamma(z) \sim \frac{(-1)^n}{\Gamma(n + 1)} \frac{1}{n + z} \quad z \sim -n \]

so we want something like

\[ A(s, t) \sim \Gamma(-t)\Gamma(-s) \]
\[ A(s, t) \sim \Gamma(-t)\Gamma(-s) \]

- to connect poles at \( t \) (or \( s \)) = 1,2,3 \ldots with physical masses use the Regge trajectory

\[ A(s, t) \sim \Gamma(n - \alpha(s))\Gamma(n - \alpha(t)) \]

- \( n,m \) determine location of first poles, e.g. \( \Gamma(-\alpha(s)) \) has, for \( \alpha(s) = 0.5 + s \), the first pole at \( s=-1/2 \) i.e. particle with imaginary mass. But \( \Gamma(1-\alpha(s)) \) has the first pole at \( s=+1/2 \) i.e. the rho-meson

- simultaneous poles in \( s \) and \( t \) (“overlapping channels”) are unexpected. To remove them use

\[ A(s, t) \sim \frac{\Gamma(n - \alpha(s))\Gamma(n - \alpha(t))}{\Gamma(n + m - \alpha(s) - \alpha(t))} \]

\( n \geq m \geq 1 \)

- what is missing is kinematical (spin) and symmetry (e.g. isospin) factors

\[ \text{it has poles and zeros!} \]
Examples:

\[ \pi \pi \rightarrow \pi \pi \]

- after isospin decomposition there are three scalar amplitudes \( A(s,t,u), B(s,t,u), C(s,t,u) \) \( \rightarrow \) Veneziano

\[ \pi N \rightarrow \pi N \]

- there are two scalar amplitudes \( A(s,t), B(s,t) \)

\[ V \pi \rightarrow \pi \pi \]

- there is one scalar amplitude \( (*) \)

**Exercise:** verify \( (*) \)
\[ A(s, t) = \frac{\Gamma(-J(s))\Gamma(-J(t))}{\Gamma(-J(s) - J(t))} \]

\[ \omega \rightarrow 3\pi \]
relativistic h.o.

QCD, loop representation, large-$N_c$, AdS/CFT, ...

$\omega \to 3\pi$

$A(s,t) = \frac{\Gamma(-J(s))\Gamma(-J(t))}{\Gamma(-J(s) - J(t))}$
Veneziano amplitude: “compact” expression for the full amplitude

\[ A(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \quad \alpha(s) = a + bs \]

\( A(s, t) \) can be written as sum over resonances in ether channel.

\[ A(s, t) = \sum_k \frac{\beta_k(t)}{k - \alpha(s)} = \sum_k \frac{\beta_k(s)}{k - \alpha(t)} \]

Note: in V-model resonance couplings, \( \beta \), are fixed! (*)

\[ \beta_k(t) \propto (1 + \alpha(t))(2 + \alpha(t)) \ldots (k + \alpha(t)) \]

**Exercise:** verify (*)
\[ V(p, \lambda) \rightarrow \pi^i(p_1)\pi^j(p_2)\pi^k(p_3) \]

\[ A(s, t, u) = \epsilon_{ijk}\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu}(p, \lambda)p_1^\nu p_2^\alpha p_3^\beta \times [A_{n,m}(s, t) + A_{n,m}(s, u) + A_{n,m}(t, u)] \]

\[ A_{n,m}(s, t) \equiv \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(n + m - \alpha_s - \alpha_t)}. \]

\[ \alpha(s) = \alpha_0 + \alpha's. \]

\[ \alpha(s) = \frac{1}{2} + s \]

Exercise: verify (*)
Resonances couplings, $\beta$, should depend on final state particles: a linear superposition of Veneziano amplitudes can be used to suppress or enhance individual resonances or trajectories

$$M = \epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha c^\beta A(s, t, u)$$

$$A = \sum_{n,m} c_{n,m} \frac{\Gamma(n - \alpha(s))\Gamma(n - \alpha(t))}{\Gamma(n + m - \alpha(s) - \alpha(t))} + (s, u) + (t, u)$$

- even-spin $\rho$'s do not couple to $\pi\pi$ and should decouple in $J/\psi \rightarrow 3\pi$
- coupling of odd-spin $\rho$'s depend on can depend vary depending on trajectory
\[ A_{n,m}(s, t) = \sum_{k}^{\infty} \frac{\beta(t)}{k - \alpha(s)} = \sum_{k}^{\infty} \frac{\beta(s)}{k - \alpha(t)} \]

**How to isolate individual poles?**

\[ n \geq m \geq 1 \]

\[ A_{1,1} = \frac{\Gamma(1 - \alpha_s)\Gamma(1 - \alpha_t)}{\Gamma(2 - \alpha_s - \alpha_t)} \]

has poles at \( \alpha_s = 1, 2, 3, \ldots \)

\[ A_{2,1} = \frac{\Gamma(2 - \alpha_s)\Gamma(2 - \alpha_t)}{\Gamma(3 - \alpha_s - \alpha_t)} \]

has poles at \( \alpha_s = 2, 3, 4, \ldots \)

\[ A_{2,2} = \frac{\Gamma(2 - \alpha_s)\Gamma(2 - \alpha_t)}{\Gamma(4 - \alpha_s - \alpha_t)} \]

has poles at \( \alpha_s = 2, 3, 4, \ldots \)

Use a linear combination of \( A_{2,1} \) and \( A_{2,2} \) to remove pole at \( \alpha_s = 2 \)

Use a linear combination of \( A_{3,1}, A_{3,2}, A_{3,3} \) to remove pole at \( \alpha_s = 3 \), etc.

\[ A_{3,1}, A_{3,2}, A_{3,3} \]

have poles at \( \alpha_s = 3, 4, 5, \ldots \)

\[ A_{4,1}, A_{4,2}, A_{4,3}, A_{4,4} \]

have poles at \( \alpha_s = 4, 5, 6, \ldots \)
\[ A_{n,m}(s, t) \rightarrow A(s, t) = \sum_{n \geq 1, n \leq m \leq 1} c_{n,m}A_{n,m}(s, t) \]

remove all poles but the one at \( \alpha = 1 \)

\[ c_{n,1} = \frac{c_{1,1}}{\Gamma(n)}, \quad c_{n,2} = -\frac{c_{1,1}}{\Gamma(n-1)}, \quad c_{n,m} = 0 \text{ for } m > 2, \]

\[ A_1(s, t) = c_{1,1} \frac{2 - \alpha_s - \alpha_t}{(1 - \alpha_s)(1 - \alpha_t)}. \]

... but the Regge limit is now lost!

remove all poles between \( N \geq \alpha \geq 2 \)

\[ A_1(s, t; N) = c_{1,1} \frac{2 - \alpha_s - \alpha_t}{(1 - \alpha_s)(1 - \alpha_t)} \times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N)\Gamma(N + 2 - \alpha_s - \alpha_t)}, \quad \text{has Regge limit is for } s > N \]
In the past this was done by choosing an arbitrary set of \( n, m \) and fitting \( c(n,m) \) to the data (e.g. Lovelace, Phys. Lett. B28, 265 (1968), Altarelli, Rubinstein, Phys. Rev. 183, 1469 (1969)).

The “new” model does this in a systematic way. In addition it allows for imaginary non-linear (and complex) trajectories without introducing “ancestors”

\[
A_n(s, t; N) = \frac{2n - \alpha_s - \alpha_t}{(n - \alpha_s)(n - \alpha_t)} \sum_{i=1}^{n} a_{n,i} (-\alpha_s - \alpha_t)^{i-1} \\
\times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N + 1 - n)\Gamma(N + n + 1 - \alpha_s - \alpha_t)}.
\]

\( n \): number of Regge trajectories
\( a_{n,i} \): determine resonance couplings
\( N \): determines the onset of Regge behavior
\( \alpha(s), \alpha(t) = \text{Re} \alpha + i \text{Im} \alpha \): with \( \text{Im} \alpha \) related to resonance widths
Different authors employed the Veneziano model for the analysis of the at-rest annihilation $\overline{N}N \rightarrow 3\pi$, using a finite number of Veneziano terms.

- Lovelace\textsuperscript{2}: a single term amplitude,
  \[ n = m = 1, \]
  \[ \alpha_s = 0.483 + 0.885s + 0.28i\sqrt{s - 4m^2} \]
- Altarelli\textsuperscript{3}: 5 terms with $n + m \leq 3$ (to reproduce the zero at $\alpha_s + \alpha_t \simeq 3$)
- Gopal\textsuperscript{4}: 5 terms with $n + m \leq 3$,
  \[ \alpha_s = 0.483 + 0.885s + iA(s - 4m^2)^B, \]
  \[ B < 1 \]

\textsuperscript{2}C. Lovelace, Phys. Lett. 25B (1968), 264
\textsuperscript{3}G. Altarelli, Phys. Rev. 183 (1969), 1469
\textsuperscript{4}G. P. Gopal, Phys. Rev. D 3 (1971), 2262

from A.Celentano
All poles below $\alpha = N$ except at $\alpha = n$

$$A_n(s, t; N) = a_{n,0} \frac{2n - \alpha_s - \alpha_t}{(n - \alpha_s)(n - \alpha_t)} \left[ \prod_{i=1}^{n-1} (a_{n,i} - \alpha_s - \alpha_t) \right]$$

$$\times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N + 1 - n)\Gamma(N + n + 1 - \alpha_s - \alpha_t)}$$

at $\alpha_s = n$ residue is a polynomial in $t$ of order $n-1$

(remember to add 1 from the Levi-Civita tensor)

$$A(s, t, u) = \epsilon_{ijk}\epsilon_{\mu \nu \alpha \beta}\epsilon_{\mu}(p, \lambda)p_1^\nu p_2^\alpha p_3^\beta$$

$$\times [A_{n,m}(s, t) + A_{n,m}(s, u) + A_{n,m}(t, u)]$$

$A_1$ has $\rho(770)$

$A_3$ has $\rho(1700)$, $\rho_3(1690)$

$A_5$ has $\rho''(2150)$, $\rho_3(2250)$, $\rho_5(2350)$
The pole in $A_{n,m}$ with residues that are polynomials in $(s,t)$ and $m$ correspond to the leading trajectory. The signature determines coupling to the pole at $s = t = 1$ trajectory. The signature of all, when studying resonance properties one is forced to work with partial waves. Proper description of resonances, however, requires that unitarity is satisfied and allows for non-linear trajectories, implementation of which decouples all, but a finite number of residues to be process dependent. One possibility is to use a linear combination of resonances alone can fix.

Since amplitudes with $n = 1$ have the lowest pole at $s = t = 1$ have to be canceled by similar poles present in amplitudes with $n > 1$ no other pole in $A_{n,m}$ with fixed $n$ can contribute to resonances by dots at integer values of spin $(s,t)$. The signature of all, when studying resonance properties one is forced to work with partial waves. Proper description of resonances, however, requires that unitarity is satisfied and allows for non-linear trajectories, implementation of which decouples all, but a finite number of residues to be process dependent. One possibility is to use a linear combination of resonances alone can fix.

Even though there are extensions of the Veneziano model forces Regge trajectories to be real and linear. Regge trajectories are non-linear, while the Veneziano model forces Regge trajectories to be real and linear.

Even though there are extensions of the Veneziano model forces Regge trajectories to be real and linear. Regge trajectories are non-linear, while the Veneziano model forces Regge trajectories to be real and linear.
Fig. 2: Dalitz plot projection of the di-pion mass distribution from $J/\psi$ decay. The solid is the result of the fit with three amplitudes and the dashed line with the amplitude $A_1$ alone. The insert shows the mass region of the $\rho_3$ and its contribution from the fit with the full set of amplitudes (solid line) as compared. Absence of the structure at 1.7GeV from the fit with the $A_1$ amplitude is indicated by the dashed line.

Fig. 3: Dalitz plot projection of the di-pion mass distribution from $\psi'$ decay. The solid is the result of the fit with three amplitudes and the dashed line is the fit with $A_1$ alone.
$B_5$ amplitude:

Reggeons/Resonances in all 5 channels

Double-Regge Exchange Limit for the $\gamma p \rightarrow K^+K^-p$ Reaction

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$B_5(s_{AB}, s_{A1}, s_{12}, s_{23}, s_{B3}) = B_4(-\alpha_{12}, -\alpha_{A1}) B_4(-\alpha_{23}, -\alpha_{B3}) \times 3F_2(\alpha_{AB} - \alpha_{12} - \alpha_{23}, -\alpha_{A1}, -\alpha_{B3}; -\alpha_{12} - \alpha_{A1}, -\alpha_{23} - \alpha_{B3})$. 
Fig. 10.3. Invariant mass distribution for $\pi^+\pi^-$ from $p\bar{n} \rightarrow \pi^+\pi^-\pi^-$. Data taken from Anninos et al. (1968). Theoretical curves are those of Lovelace (1969b) and Berger (1969a).

$K^-p \rightarrow \pi^-\pi^+\Lambda$ in Veneziano Model

The columns are, from left to right, for $\pi^-p \rightarrow \pi^-\pi\Lambda$, $\pi^-p \rightarrow \pi^-\pi\Lambda$, $\pi^-p \rightarrow \pi^-\pi\Lambda$. 

- 5334 events
- 3097 events
- 1195 events

$\pi^0p \rightarrow \pi^0\pi^0n$ at 6 GeV/c