Dispersion theory in hadron form factors

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Objectives

• Review hadron interactions with electroweak fields: currents, matrix elements, form factors

• Apply methods of amplitude analysis to form factors: Analyticity, dispersion relations, unitarity

• Learn about applications to actual experimental data

• Outline connections with other fields: QCD and partonic structure
Motivation

• Hadron reactions ↔ structure

• Simple application of amplitude analysis concepts: Single-variable function $F(t)$

• Great practical use

• Experimental programs: Electron scattering JLab, Mainz, MIT, atomic physics with electronic/muonic hydrogen PSI, $e^+e^-$ annihilation BES, KLOE, BABAR, etc., $p\bar{p}$ PANDA SM/BSM processes with hadron production, neutrino interactions with matter

• Connection with QCD: Generalized parton distributions, Lattice QCD methods
Outline

1 – Hadron interactions with external fields
Currents, matrix elements, form factors
Spacelike and timelike FFs
Elastic scattering and annihilation

2 – Analytic properties of form factors
Analytic $F(t)$, physical sheet, principal cut
Dispersion relations and convergence
Spectral function and $t$-channel processes
Methods for constructing spectral function

3 – Spectral functions from amplitude analysis
Unitarity relation for $\pi\pi$ cut
$\pi\pi \to NN$ partial wave amplitudes from $\pi N \to \pi N$ data
Pion form factor from $e^+e^- \to \pi^+\pi^-$ data

4 – Nucleon electromagnetic form factors
Evaluation of dispersion integral
Comparison with spacelike FF data
Open questions

5 – Extensions and connections
Nucleon scalar form factor
Nucleon axial form factors
Transverse densities and generalized parton distributions
EM interactions: Hadrons in external field

- External field — electromagnetic/weak/other
  - Accelerate existing hadron
  - Create/annihilate hadron-antihadron pair

- Interaction described by Lagrangian density
  \[ L_{\text{em}}(x) = e A_\mu(x) J^\mu(x), \quad \text{current operator} \]

- Transition matrix elements
  \[ \langle h(p') | J^\mu(x) | h(p) \rangle, \quad \langle h(p') \bar{h}(p) | J^\mu(x) | 0 \rangle \]
  Treat as abstract objects, cf. hadron scattering amplitudes
EM interactions: Lepton scattering

\[ M(lh \rightarrow l'h') = e^2 \langle h(p')|J^\mu(0)|h(p)\rangle \quad D_{\mu\nu}(k - k') \quad \langle l(k')|j^\nu(0)|l(k)\rangle \]

\[ M(l^+l^- \rightarrow hh) = e^2 \langle h(p')\bar{h}(p)|J^\mu(0)|0\rangle \quad D_{\mu\nu}(k + k') \quad \langle 0|j^\nu(0)|l(k')l(k)\rangle \]

- External field excited by lepton scattering
- Lepton current and Green function given by QED (EW theory)
- Explicit form of amplitude and cross section

→ Exercise
EM interactions: Current matrix element

- Translational invariance

\[ J_\mu(x) = e^{ipx} J_\mu(0) e^{-ipx} \]

finite translation operator

\[ \langle h(p')| J_\mu(x)| h(p) \rangle = e^{i(p'-p)x} \langle h(p')| J_\mu(0)| h(p) \rangle \quad x\text{-dep explicit} \]

- Current conservation

\[ \partial_\mu J_\mu(x) = 0 \rightarrow \partial_\mu \langle h(p')| J_\mu(x)| h(p) \rangle = 0 \]

\[ (p' - p)_\mu \langle h(p')| J_\mu(0)| h(p) \rangle = 0 \quad \text{transversality} \]

- Similar relations for \( \langle h(p')\bar{h}(p)| J_\mu(x)|0 \rangle \) with \( p \rightarrow -p \)
**EM interactions: Form factor spin-0 hadron**

- **Structural decomposition**

\[
\langle h(p')|J^\mu(0)|h(p)\rangle \leftrightarrow \text{structures}
\]

Lorentz 4-vector:

\[
\langle |J^\mu| \rangle \propto (p' + p)^\mu, \ (p' - p)^\mu
\]

Current conserved:

\[
(p - p')_\mu \langle |J^\mu| \rangle = 0
\]

\[
\langle |J^\mu| \rangle = (p' + p)^\mu F(t), \quad t \equiv (p' - p)^2
\]

- **\( F(t) \)** invariant form factor, cf. invariant amplitudes

Physical region for scattering: \( t < 0 \)

\( F(0) = \) hadron charge (in units of \( e \))
• Similar decomposition

\[ \langle h(p')\bar{h}(p)|J^\mu(0)|0\rangle = (p' - p)^\mu F(t), \quad t = p' + p \]

Physical region for \( h\bar{h} \) creation: \( t > 4M_h^2 > 0 \)

• Crossing symmetry: Scattering and annihilation matrix elements are described by a single analytic function

\[ F(t), \quad \left\{ \begin{array}{ll}
  t < 0, & \text{scattering} \quad \text{“spacelike FF”} \\
  t > 4M_h^2, & \text{annihilation} \quad \text{“timelike FF”}
\end{array} \right. \]
EM interactions: Form factors spin-1/2 hadron

- Lorentz invariance requires
  \[ \langle h(p', \lambda') | J^\mu(0) | h(p, \lambda) \rangle = \bar{u}(p', \lambda') \Gamma^\mu u(p, \lambda) \]
  Bilinear form in hadron bispinors \( u, \bar{u} \)

- Independent structures in \( \Gamma^\mu \)?
  
  Heuristic approach: Use transversality, Dirac eqn, gamma matrix identities
  
  Systematic approach: Count helicity amplitudes

  \[
  \Gamma^\mu = \gamma^\mu F_1(t) + \frac{i\sigma^{\mu\nu}(p' - p)^\nu}{2M_h} F_2(t)
  \]

- \( F_{1,2}(t) \) invariant FFs
  \[
  F_1(0) = \text{charge}, \quad F_2(0) = \text{anomalous magnetic moment}
  \]
  
  Dirac and Pauli FFs; also other choices \( G_E, G_M \)
**Analytic properties: Form factor**

- *FF analytic function of $t$*
  
  Physical sheet, approached from $t < 0$
  
  No singularities at $t < 0$

- *Singularities at $t > 0$ — poles, cuts*
  
  Result from “processes”: current $\rightarrow$ hadronic state $\rightarrow h\bar{h}$
  
  Can occur below the physical $h\bar{h}$ threshold
Analytic properties: Examples

- Nucleon electromagnetic FFs, isovector

\[ 4M^2_\pi \quad 16M^2_\pi \quad 4M^2_K \quad 4M^2_N \]

(unphysical) \rightarrow \rightarrow (physical)

- Pion electromagnetic FF

\[ 4M^2_\pi \quad 16M^2_\pi \quad 4M^2_K \quad 4M^2_N \]

(physical) \rightarrow \rightarrow (physical)
Analytic properties: Dispersion relation

\[ F(t) = \int_{t_{\text{thr}}}^{\infty} \frac{dt'}{\pi} \frac{\text{Im} F(t + i0)}{t' - t} \]

Dispersion relation

• Convergence at \(|t| \to \infty\): \( F(t) \sim t^{-1} \) (pion), \( \sim t^{-2,3} \) (nucleon) from QCD

• DR represents FF at all \( t \) on physical sheet

    Implements correct analytic properties. Useful tool

• Spectral function \( \text{Im} F(t + i0) = [F(t + i0) - F(t - i0)]/(2i) \)

    Need to know it in order to evaluate integral. Unphysical region!
Analytic properties: Spectral functions

- Amplitude analysis techniques + hadronic scattering data: Unitarity, analytic continuation
  

- Fits to spacelike FF data
  
  Hohler et al., Nucl. Phys. B 114 (1976) 505

- Dynamical calculations: Chiral effective field theory
  

- Combination of above methods
  
  $\chi$EFT + unitarity: Granados, Leupold, Perotti, Eur. Phys. J. A 53, 117 (2017);
Analysis: Unitarity relation

\[ F_i = \Gamma_i \Gamma_i^* \]

\[ t > 4M_{\pi}^2 \]

\[ I = J = 1 \]

\[ \Gamma_i^i(t) = \frac{k_{\text{cm}}^3}{\sqrt{t}} \Gamma_i(t) F_{\pi}^*(t) \]

- At \( 4M_{\pi}^2 < t < 16M_{\pi}^2 \) only \( \pi\pi \) channel open — two-pion cut

- \( \text{Im} F_i(t) \) from elastic unitarity relation

CM frame of timelike process, \( k_{\text{cm}} = \sqrt{t/4 - M_{\pi}^2} \) pion CM momentum

\( F_{\pi}(t) \) current \( \rightarrow \pi\pi \) partial-wave amplitude = pion timelike FF

\( \Gamma_{\pi}(t) \) \( \pi\pi \rightarrow N\bar{N} \) partial-wave amplitude

Amplitudes \( F_{\pi}(t) \) and \( \Gamma_{\pi}(t) \) have same phase — Watson theorem

Amplitudes "contain" \( \rho \) as \( \pi\pi \) resonance

Here: Nucleon isovector EM FFs
Analysis: $\pi\pi \rightarrow N\bar{N}$ partial-wave amplitudes

**Idea:** Construct $\pi\pi \rightarrow N\bar{N}$ PWAs $\Gamma_i(t)$ from $\pi N \rightarrow \pi N$ scattering data using amplitude analysis techniques.

Two major challenges:

- $t$–channel partial wave projection requires knowledge of amplitude in unphysical region of $s, u$–channel processes $s, u < (M_\pi + M_N)^2$.
  
  → analytic continuation in $s, u$ at fixed $t$.

- For spectral function we need $t$-channel PWAs at $t > 4 M_\pi^2 > 0$, while $\pi N \rightarrow \pi N$ data are at $t < 0$.
  
  → analytic continuation in $t$. 
Analysis: $\pi \pi \rightarrow N \bar{N}$ partial-wave amplitudes

- Mandelstam diagram

$$\nu = \frac{s - u}{4M_N}$$

crossing-symmetric variable
Analysis: $\pi\pi \rightarrow N\bar{N}$ partial-wave amplitudes

I) Construct $\pi N \rightarrow \pi N$ invariant amplitudes $A_{\pm}$ with proper analyticity in $s, u$ at fixed $t < 0$

Data in physical region $s, u > (M_\pi + N_N)^2$

Continue to unphysical $s, u < (M_\pi + N_N)^2$

II) Calculate $t$-channel partial-wave projections

$$f_{\pm}(t) = \text{(factor)} \times \int_{-1}^{1} dz \left\{ \frac{P_1(z)}{P_0(z) - P_2(z)} \right\} A_{\pm}(\nu, t) \leftrightarrow \Gamma_{1,2}(t)$$

$$\nu = \frac{s - u}{4M_N}, \quad z = -\cos \theta_t = \frac{M_{N\nu}}{\sqrt{M_N^2 - t/4} \sqrt{M_\pi^2 - t/4}}$$

III) Analytically continue PWAs to region $t > 4M_\pi^2$

Use DR for PWA (left-hand cut), $N/D$ method, estimate uncertainties

Hohler et al 74
Analysis: $\pi\pi \rightarrow N\bar{N}$ partial-wave amplitudes

- What hadronic processes contribute to the $\pi N \rightarrow \pi N$ amplitude?

- Born term amplitudes with intermediate $N, \Delta$

  Singularities at $s, u = M_N^2, M_\Delta^2$

  Contribute to left-hand cut of PWA

- Also other intermediate states $\pi N$ etc.
| Analysis: Pion timelike form factor |

Figure from Jegerlehner 15

- \(|F_\pi(t)|^2\) measured in \(e^+e^- \rightarrow \pi^+\pi^-\) exclusive annihilation

\(\rho\) as \(\pi\pi\) resonance

- Phase (Im/Re) determined by fit to resonant amplitude parametrization

Gounaris-Sakurai: Effective range expansion of \(\pi\pi\) phase shift,
good description of line shape  

Gounaris, Sakurai, PRL 21 (1968) 244
Analysis: Spectral function

- Spectral function on $\pi\pi$ cut calculated

  Elastic unitarity condition
  + data $\pi N \rightarrow \pi N$, $e^+ e^- \rightarrow \pi^+ \pi^-$
  + analytic continuation

  Valid up to $t \sim 1$ GeV$^2$

  Contains $\rho$ resonance

- Alt. approach: Dynamical calculations

  $\chi$EFT + N/D method

**Nucleon FF: Dispersion integral**

- Evaluate dispersion integral
  \[ F_i(t) = \int_{4M_n^2}^{\infty} \frac{dt'}{\pi} \frac{\text{Im } F_i(t + i0)}{t' - t} \]

- What about higher \( t' \)?

- Constraints on spectral function
  \[ F_1(t) \sim t^{-2} \rightarrow \int_{4M_n^2}^{\infty} \frac{dt'}{\pi} \text{Im } F_i(t') = 0 \]
  asymptotic behavior

  \[ F_1(0) = \text{charge} = \int_{4M_n^2}^{\infty} \frac{dt'}{\pi} \frac{\text{Im } F_i(t')}{t'} \] charge

- Parametrize isovector spectral function at high \( t' \): Simple pole, multiple poles
  Isoscalar: \( \omega, \phi + \text{high } t' \text{ poles} \)

  Hohler et al NPB 114 505 (1976); Belushkin et al PRC 75 035202 (2007).
Nucleon FF: Fits to spacelike data

- High–$t'$ spectral functions determined by fit to spacelike FF data
  Belushkin et al PRC 75 035202 (2007); Lorenz et al EPJA 48, 151 (2012)

- Good description of FF data
  Analytic parametrization of FF. Permits $t \to 0$ extrapolation, derivatives
Nucleon FF: Derivatives

\[ F(t) = \int_{4M_{I\pi}^2}^{\infty} \frac{dt'}{\pi} \frac{\text{Im} F_i(t + i0)}{t' - t} \quad \rightarrow \quad F'(t = 0) = \int_{4M_{I\pi}^2}^{\infty} \frac{dt'}{\pi} \frac{\text{Im} F_i(t + i0)}{(t')^2} \]

- Derivatives of FFs at \( t = 0 \) given by well-convergent integrals
  - Dominated by \( t' < 1 \text{ GeV}^2 \). Predictions of unitarity!
    - "\( \rho \) meson dominance"

- Nucleon charge/magnetic radii

\[ \langle r^2 \rangle_1 = 6 F'_1(0) \quad \quad \langle r^2 \rangle_2 = 6 \frac{F'_2(0)}{F_2(0)} \]

Interpretation in context of non-relativistic systems:
Radius of 3D charge/magnetization density in Breit frame \( q^\mu = (0, q) \)

Relativistic systems: Use 2D transverse densities!
Amplitude analysis methods described here can be applied to variety of meson and baryon form factors

Scalar nucleon FF

\[ O_\sigma(x) = \sum_{u,d} m\bar{\psi}(x)\psi(x), \quad \langle N(p')|O_\sigma(0)|N(p)\rangle = \bar{u}'u\sigma(t) \]

Fundamental interest: Mass term of QCD Hamiltonian, trace of QCD energy-momentum tensor, "\(\sigma\) term"

Two-pion cut, unitarity condition. Coupling to \(K\bar{K}\) at \(t > 4M_K^2\)

Pion scalar FF from dispersion theory with \(\chi\)EFT input
Extensions: Vector and axial FFs

- **Isoscalar vector FF**

  Intermediate states $3\pi, K\bar{K} +$ higher

  3-body unitarity: Techniques being developed
  JPAC: Pilloni, Szczepaniak

  Coupled channel problem

- **Isovector pseudoscalar and axial FF**

  $J^{5\mu}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\ [u - d]$

  Pion pole in induced pseudoscalar FF

  $3\pi +$ higher intermediate states in axial FF
Structure of relativistic system is naturally described at fixed light-front time $x^+ = x^0 + x^3$

- Boost-invariant definition of “time”
- Frame-independent wave functions, densities
- Unambiguous “particle content” of system

- FFs expressed through transverse densities
  
  \[ F_{1,2}(t = -\Delta_T^2) = \int d^2b \ e^{i\Delta_T b} \rho_{1,2}(b) \]

Cumulative charge/magnetization density at transverse position $b$

Connection with GPDs in QCD

\[ \rho_1(b) = \sum_q e_q \int_0^1 dx \ [q - \bar{q}](x, b) \]
Extensions: Transverse densities

- Dispersive representation

\[ \rho(b) = \int_0^\infty \frac{dt}{2\pi} K_0(\sqrt{tb}) \frac{\text{Im} F(t)}{\pi} \]

\[ K_0 \sim e^{-b\sqrt{t}} \] ensures exponential suppression of large \( t \)

Distance \( b \) selects \( \sqrt{t} \sim 1/b \): Filter for spectral function

Peripheral densities dominated by lowest–mass intermediate states

Strikman, Weiss PRC 82 042201 (2010); Miller, Strikman, Weiss PRC 84 045205 (2011)
Extensions: Transverse densities

- Spatial structure of $t$-channel processes

  Peripheral $b \gtrsim 1 \text{ fm} \iff \pi\pi$ intermediate state (includes $\rho$)

  Central $b \ll 1 \text{ fm} \iff$ higher-mass intermediate states

- Quantify “range” of exchange mechanisms
- Spectral analysis of isovector charge density

Near-threshold $\pi\pi$ relevant only at $b > 3$ fm

Intermediate $b = 0.5 - 1.5$ fm dominated by $\rho$

Higher-mass states relevant only at $b < 0.3$ fm
- Spectral analysis of isoscalar density

\[ \omega \text{ dominates at } b > 1.5 \text{ fm} \]

Large cancellations between \( \omega \) and higher-mass states at \( b = 0.5 - 1 \text{ fm} \)
Extensions: Transverse densities

- Quark-hadron duality in transverse densities
  
  Hadron exchanges in $t$–channel $\leftrightarrow$ Partonic configurations in $s$–channel

- Pion FF and transverse density
  
  $|F_{\pi}(t)|^2$ from $e^+e^- \rightarrow \pi^+\pi^-$ data

  Spectral function $\text{Im } F_{\pi}(t)$ from fit to resonant amplitude parametrization

  FF and transverse density from dispersion relation

- Resonance transition FFs and densities
  
  $N \rightarrow N^*$ and $N^* \rightarrow N^*$ transition FFs

  S-matrix theory: Resonance structure defined at complex pole
Extensions: Pion FF and density

- $|F_\pi(t)|^2$ measured in $e^+e^- \rightarrow \pi^+\pi^-$ up to $t \sim 10$ GeV$^2$
- $\text{Im} F_\pi(t)$ from resonant amplitude fit
- Transverse density from dispersion relation
- High density in center of pion: Pointlike $q\bar{q}$ configurations

Miller, Strikman, Weiss, PRD 83, 013006 (2011)
Summary

• Interaction of hadrons with external fields described by current matrix elements

• FFs have analytic properties similar to hadronic scattering amplitudes

• Spectral functions on $\pi\pi$ cut from elastic unitarity and hadronic data

• Good description of spacelike nucleon FFs and derivatives

• Transverse densities connect hadronic exchanges with partonic structure

• Amplitude analysis is not just science, but also art!