Classwork

Wigner $D$-Matrices

The Wigner matrices of the rotation group are given as

$$D^{(j)}_{\sigma'\sigma}(\alpha, \beta, \gamma) = e^{-i\alpha\sigma' - i\gamma\sigma}d^{(j)}_{\sigma'\sigma}(\beta)$$

(1)

where $\alpha$, $\beta$, and $\gamma$ are the Euler angles (see Fig. 1) and the little $d$-matrices are defined by

$$d^{(j)}_{\sigma'\sigma}(\beta) \equiv \langle j, \sigma' | e^{-i\beta J_y} | j, \sigma \rangle.$$  

(2)

Using the properties of the rotation generators:

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

(3)

$$J_{\pm} = J_x \pm iJ_y,$$

(4)

$$J_z | j, \sigma \rangle = j(j + 1) | j, \sigma \rangle,$$

(5)

$$J_\pm | j, \sigma \rangle = \sqrt{j(j + 1) - \sigma(\sigma \pm 1)} | j, \sigma \pm 1 \rangle,$$

(6)

$$J_\pm | j, \sigma \rangle = \sqrt{j(j + 1) - \sigma(\sigma \pm 1)} | j, \sigma \pm 1 \rangle,$$

(7)

derive the little $d$-matrices for the spin-1/2 representation and the spin-1 representation. If we define the half-angle factor $\xi_{\sigma'\sigma}(z) = \sqrt{(1 + z)/2}^{\sigma' + \sigma} \sqrt{(1 - z)/2}^{\sigma' - \sigma}$, write the $d$-matrices in the form $d^{(j)}_{\sigma'\sigma}(z) = \xi_{\sigma'\sigma}(z)P^{(j)}_{\sigma'\sigma}(z)$, where $z = \cos \beta$, and $P^{(j)}_{\sigma'\sigma}(z)$ is a regular polynomial in $z$.

Figure 1: Euler Angles.