Partial Waves of the Born Amplitude

Consider the $2 \to 2$ process $aa \to bb$, with $m_a < m_b$, where the $a$ and $b$ particles are pseudoscalars ($J^P = 0^-$). The process includes a $t$-channel exchange of particle $c$, which is a scalar meson ($J^P = 0^+$), with mass $M$. The amplitude is of the form

$$A(s,t) = \frac{g^2}{t - M^2},$$

where $g$ is the coupling between the $a$, $b$, and $c$ particles (see Fig. 1). Work in the center-of-momentum system (CMS).

### 3.1 $s$-channel

(1) Write $t$ in terms of the masses, $s$, and $z_s = \cos \theta_s$, where $\theta_s$ is the scattering angle in the $s$-channel CMS.

(2) Write the allowed $J^P$ quantum numbers of the ($bb$) two-particle system.

(3) The partial wave expansion is defined as

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell(s) P_\ell(\cos \theta_s),$$

where $P_\ell$ are the Legendre functions of the 1st kind. Write the $S$-wave partial wave amplitude of the Born amplitude Eq. (1).

(4) Write the partial wave amplitudes $a_\ell(s)$ of the Born amplitude Eq. (1) in terms of the Legendre functions of the 2nd kind,

$$Q_\ell(z) = \frac{1}{2} \int_{-1}^{+1} dz' P_\ell(z'),$$

(5) For $s \to s_{th}^b = 4m_b^2$, what is the behavior of $a_\ell(s)$?

(6) Where do the branch cuts occur for the $\ell = 0$ partial wave? Draw the branch cuts for $a_{\ell=0}(s)$ in the complex $s$-plane.
3.2 \textit{t}-channel

Now consider the \textit{t}-channel process of the above reaction ($ab \rightarrow \bar{a}\bar{b}$). The \textit{t}-channel partial wave expansion is

$$\mathcal{A}(s, t) = \sum_{L=0}^{\infty} (2L + 1) a_L(t) P_L(\cos \theta_t),$$

where $\theta_t$ is the scattering angle in the \textit{t}-channel CMS. Find the \textit{t}-channel partial wave amplitudes $a_L(t)$ for the Born amplitude Eq. (1).

![Diagram for Born amplitude in (1).](image)

Properties of $P_\ell$

In the following we consider only $\ell$ integer. The first few $P_\ell$ functions are (see Fig. 2)

$$P_0(z) = 1$$
$$P_1(z) = z$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1).$$

In general, all $P_\ell(z)$ can be found by the recursion relation

$$P_0(z) = 1, \quad P_1(z) = z$$
$$P_\ell(z) = z(2 - \ell^{-1})P_{\ell-1}(z) - (1 - \ell^{-1})P_{\ell-2}(z) \text{ for } \ell > 1.$$  

The functions are orthogonal

$$\int_{-1}^{+1} dz P_\ell(z) P_{\ell'}(z) = \frac{2}{2\ell + 1} \delta_{\ell\ell'}.$$
Properties of $Q_\ell$

In the following we consider only $\ell$ integer. The first few $Q_\ell$ functions are (see Fig. 3)

\begin{align*}
Q_0(z) &= \frac{1}{2} \ln \left( \frac{1 + z}{1 - z} \right) \quad \text{(11)} \\
Q_1(z) &= \frac{z}{2} \ln \left( \frac{1 + z}{1 - z} \right) - 1 \quad \text{(12)} \\
Q_2(z) &= \frac{3z^2 - 1}{4} \ln \left( \frac{1 + z}{1 - z} \right) - \frac{3z}{2} \\&\quad \text{(13)}
\end{align*}

In general, all $Q_\ell(z)$ can be found by the recursion relation

\begin{align*}
Q_0(z) &= \frac{1}{2} \ln \frac{z + 1}{z - 1}, \quad Q_1(z) = zQ_0(z) - 1 \quad \text{(14)} \\
Q_\ell(z) &= z(2 - \ell^{-1})Q_{\ell-1}(z) - (1 - \ell^{-1})Q_{\ell-2}(z) \quad \text{for } \ell > 1. \quad \text{(15)}
\end{align*}

The asymptotic behavior of the $Q_\ell$ function is

\begin{align*}
Q_\ell(z) &\to \frac{\pi^{1/2}}{(2z)^{\ell+1} (\ell + \frac{1}{2})!} \quad \text{as } z \to \infty. \quad \text{(16)}
\end{align*}

Note the completeness condition

\begin{align*}
\sum_{\ell=0}^{\infty} (2\ell + 1) Q_\ell(z') P_\ell(z) = \frac{1}{z' - z}. \quad \text{(17)}
\end{align*}
Figure 3: $Q_\ell$ for $\ell = 0, 1, 2$. 