1. Motivation and definition
2. SME related cases: Flat-space limit
3. Curved space example -- b-space
4. Some notes on H space (in preparation)

Classical Kinematics for Lorentz Violation, Kostelecký, NR,

Riemann-Finsler Geometry and Lorentz-Violating Kinematics, Kostelecký,

Neil Russell
Northern Michigan University

IUCSS Summer School on the Lorentz- and CPT-violating Standard-Model Extension
Indiana University, Bloomington, June 3-9, 2012
The interval of a path: \( S' = \int_{A}^{B} d\tau \)

What path is the ‘shortcut’?

The solutions are geodesics

\[
\dot{x}^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta
\]
Lagrangian function

Using metric $g_{\mu\nu}$ with signature $(-, +, +, + \ldots)$

"pseudo-Riemann metric"

\[
d\tau = \sqrt{-x^2} \, d\lambda = \sqrt{-\frac{dx^\mu}{d\lambda} \, g_{\mu\nu}(x) \frac{dx^\nu}{d\lambda}} \, d\lambda = \sqrt{-u^\mu \, g_{\mu\nu}(x) \, u^\nu} \, d\lambda
\]

So action $S = \int_{A \rightarrow B} \sqrt{-g(x) \, u} \, d\lambda$

$L = \int_{A \rightarrow B} L(x, u) \, d\lambda$

$x^\mu = \text{position}$

$u^\mu = \text{velocity}$
i) We need timelike curves to ensure

\[-uu \geq 0\]

ii) The action $S$ is independent of the choice of curve parameter $\lambda$. This can be ensured if $L$ is homogeneous of degree 1 in $u^\mu$.

ie if $L(x, ku) = kL(x, u), \quad k > 0$.

Equivalently: $\frac{\partial L}{\partial u^\mu} u^\mu = L$ Euler's theorem

$\Rightarrow L = -p_\mu u^\mu$ canonical momentum $p_\mu$.

iii) $L \geq 0$ since we take the positive square root.
(iv) Tangent spaces in pseudo-Riemann geometry are isotropic.

No preferred spacetime direction

⇒ Local Lorentz invariance
(v) We can recover the metric $r_{\mu \nu}(x)$ from $L(x, u)$:

Differentiate once,

$$\frac{2}{\partial u^\nu} L^2 = \frac{2}{\partial u^\nu} (-u u^\nu) = -2 u^\mu r_{\mu \nu}$$

twice:

$$\frac{2}{\partial u^\mu} \frac{2}{\partial u^\nu} L^2 = \frac{2}{\partial u^\mu} (-2 u^\mu r_{\mu \nu}) = -2 r_{\mu \nu}$$

$$\implies \quad r_{\mu \nu}(x) = -\frac{1}{2} \frac{2}{\partial u^\mu} \frac{2}{\partial u^\nu}(L^2)$$

Why not...

\[ M, \ r_{\mu \nu}(x) \] 
\[ \text{REPLACE} \] 
\[ M, \ L(x, u) \]
Explicit symmetry breaking

Obtaining $r^\mu_\nu$ from a Lagrange function suggests a method for introducing explicit symmetry breaking:

→ introduce a vector field $a_\mu(x)$
to the manifold

(* or a tensor field)

→ add it to the Lagrange function as a scalar, e.g.

$$L(x, u) = \sqrt{-uu^\nu} + a_\mu u^\mu$$

→ The derived metric $g^\mu_\nu(x, u) \equiv -\frac{1}{2} \frac{2}{\partial u^\mu} \frac{2}{\partial u^\nu}(L^2)$

→ Riemann–Finsler geometry contains these ideas
Definition

Manifold $M$ with points $x \in M$.

"Smooth:" i.e. $C^\infty$ around.

Vector $y$ at point $x$ lies in tangent space $T_x M$ at $x$.

Vector magnitudes, angles set by Riemann metric $g_{jk}(x)$.

"Finsler structure" a real-valued function $F(x,y)$ such that

(a) Non-negative: $F(x,y) \geq 0$

(b) Smooth: $F$ is $C^\infty$ on $TM \setminus \{\text{excluded slits}\}$

(c) Homogeneous of degree 1 in $y$. $F(x,\lambda y) = \lambda F(x,y) \quad \forall \lambda > 0$

(d) Positive definite $g_{jk}: g_{jk} = \frac{1}{2} \partial_y y_j \partial_y y_k F^2$

Finsler manifold: a manifold $M$ with a Finsler structure $F$.
<table>
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<th>Notation and terminology</th>
<th>Riemann–Finsler</th>
<th>pseudo-Riemann Finsler</th>
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<tr>
<td>Dimension</td>
<td>( n )</td>
<td>( n+1 )</td>
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<tr>
<td>coordinates</td>
<td>( x^j ) ( j = 1, \ldots, n )</td>
<td>( x^\mu ) ( \mu = 0, 1, \ldots, n )</td>
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<tr>
<td>velocity</td>
<td>( y^j = \frac{dx^j}{d\lambda} )</td>
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<td>structure</td>
<td>( F(x,y) )</td>
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<td>underlying metric</td>
<td>( r_{ijk} ) ( \text{pos def.} )</td>
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<td>metric</td>
<td>( g^{jk} )</td>
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<td>norm</td>
<td>( | y | = \sqrt{y^j r_{jk} y^k} = \sqrt{y^2} )</td>
<td>For timelike ( u ), ( | u | = \sqrt{-u^2} )</td>
</tr>
</tbody>
</table>

**Properties**

- all vectors are "timelike"
  - \( y^2 > 0 \)
- distinct categories
  - timelike ±
  - spacelike ±
  - lightlike ±

\[ u^2 < 0 \]
Homogeneity

Some points

Euler's theorem: \( F \) homog of degree \( n \)

\[
y^{\frac{\partial}{\partial y_j}} F(y) = n F(y)
\]

Results include

* \( \partial y_j F \) reduces homogeneity by 1
* \( F \) of degree \( n \), \( G \) of degree \( m \)

\[
\Rightarrow FG \text{ of degree } n + m
\]

Example functions using \( \vec{y} = (y, y') \)

\[
F(y, y') = \sqrt{y^2 - 3yy' + y'^2} \quad \text{has degree 1}
\]

\[
F(y, y') = \frac{y}{y'} \quad \text{has degree 0}
\]

Exercise

Show that \( F = \sqrt{y g yy'} \) \( (F \text{ has degree 1}) \)

\[
g_{jk} = \frac{1}{2} \partial y_j \partial y_k F^2
\]
Solution

\[ y^g y^j = \frac{1}{2} y^k y^j \partial_y \left[ \partial_{y_k} F^2 \right] \quad \text{degree 1} \]

\[ = \frac{1}{2} y^k \partial_{y_k} \left[ F^2 \right] \quad \text{degree 2} \]

\[ = \frac{1}{2} \partial_{y_k} F^2 \]

\[ = \frac{1}{2} 2 F^2 \]

\[ \Rightarrow y^g y^j = F^2 \]

\[ \Rightarrow \sqrt{y^g y^j} = F \quad (g_{jk} \text{ is pos. def.}) \]
Which Finsler structures are related to the SME?

Curvature:
- $L = \sqrt{(b \cdot u)^2 - b^2 u^2}$
- $L = b \mu e^a \delta_5 \delta^a$

Curved

Flat

Curvature relation:
- $\eta_{\mu \nu} = r_{\mu \nu}(x)$
- $\eta_{\mu \nu} \rightarrow r_{\mu \nu}(x)$

Dispersion relation:
- Free-particle wave packet
- Classical free particle
On a small enough region, the background is uniform.

$\Rightarrow$ Minkowski SME
Dispersion relation

Idea: seek a classical Lagrange function $L(x,u)$ producing the same dispersion relation as that for the fermion sector.

Conventional dispersion is $p^2 = m^2$, quadratic in $p^\nu$, can write $E^2 = m^2 + \vec{p}^2$ or $R(m; p_\nu) = 0$

Full result for minimal SME terms is known:

$$R(m, a_\nu, b_\nu, c_{\mu\nu}, d_{\mu\nu}, e_\nu, f_\nu, H_{\mu\nu}, g_{\mu\nu}; p_\nu) = 0$$

Special cases

$$(p - a)^2 - m^2 = 0$$

$$(-p^2 + b^2 + m^2)^2 - 4(b \cdot p)^2 + 4(b \cdot p)^2 = 0$$

$$p(\delta + 2c + c^T c)p - m^2 = 0$$

Kostelecký and Lehnert, PRD 63, 065008 (2001)
Method for finding $L(x, u, \lambda)$

Note the following requirements on $L(x, u, \lambda)$

a) Energy and momentum conservation in flat space
   $\Rightarrow L$ can't depend on $x^\mu$

b) $L$ must be independent of the choice of path parameter
   $\Rightarrow L$ can't depend on $\lambda$

   and, $L = \frac{\partial L}{\partial u^\mu} u^\mu \Rightarrow L = -p^\mu u^\mu$ (homogeneity)

Conditions derived from the dispersion relation:

$$R(m_j, a_\nu, b_\nu \ldots ; p^\mu) = 0$$

c) Group velocity $\partial p_\nu / \partial p_k$ must match classical velocity $u^k / u^0$

$$u^k / u^0 = -\frac{\partial p_\nu}{\partial p_k} \quad k = 1, 2, 3$$

Five algebraic equations in 9 variables: $L, p^\mu, u^\mu$.

Use 4 to eliminate $p_\nu$: $P(L, u^\mu, m_j, a_\nu, b_\nu \ldots) = 0$

Roots $L(u)$ of polynomial $P$ are candidate Lagrange functions.
Conventional free particle

\[ \vec{p} \quad \text{energy} \]

\[ m \]

Dispersion relation: \[ p_0^2 - p_1^2 = m^2 \] (in 1+1 space) \((x^0, x^1)\)

\[ \Rightarrow \text{Find } L(u^0, u^1) \]

Additional conditions:

\[ \frac{u^1}{u^0} = -\frac{\partial p_0}{\partial p_1} \quad \text{velocity} \]

\[ L = -p_0 u^0 - p_1 u^1 \quad \text{homogeneity} \]

Counting: 3 equations, 5 variables \((u^0, u^1, p_0, p_1, L)\)

\[ \Rightarrow 1 \text{ equation, 3 vars } (u^0, u^1, L) \]
Find $L(u^0, u^1)$ for a conventional particle with dispersion relation $\rho_0^2 - \rho_1^2 = m^2 \quad (1)$ (in 1+1 space)

**Solution:**

**Homogeneity:** $L = -\rho_0 u^0 - \rho_1 u^1 \quad (2)$

**Group velocity:** $\frac{u^1}{u_0} = -\frac{\partial \rho_0}{\partial \rho_1} \quad (3)$

Implicit differentiation of (1) wrt $\rho$:

$$2\rho_0 \frac{\partial \rho_0}{\partial \rho} - 2\rho_1 = 0 \Rightarrow \frac{\partial \rho_0}{\partial \rho} = \frac{\rho_1}{\rho_0}$$

$\Rightarrow$ eq. (3) becomes $\frac{u^1}{u_0} = -\frac{\rho_1}{\rho_0} \quad (3')$

Use (3') to eliminate $\rho_1$ from (2)

$$L = -\rho_0 u^0 - \left(-\frac{u^1}{u_0 \rho_0}\right) u^1 \Rightarrow L = -\rho_0 u_0 \left[1 - \left(\frac{u^1}{u_0}\right)^2\right] \quad (2')$$
Use (3') to eliminate $p_1$ from (1):

$$p_0^2 - \left( - \frac{u_1'}{u_0} p_0 \right)^2 = m^2$$

$$\Rightarrow p_0^2 \left[ 1 - \left( \frac{u_1'}{u_0} \right)^2 \right] = m^2$$

$$\Rightarrow p_0 = \pm \frac{m}{\sqrt{1 - \left( \frac{u_1'}{u_0} \right)^2}} \quad -(1')$$

Use (1') to eliminate $p_0$ from (2')

(2') : $L = -p_0 u_0 \left[ 1 - \left( \frac{u_1'}{u_0} \right)^2 \right]$  

$$= \pm \frac{m u_0}{\sqrt{1 - \left( \frac{u_1'}{u_0} \right)^2}} \left[ 1 - \left( \frac{u_1'}{u_0} \right)^2 \right]$$

$$\Rightarrow L = \pm m \sqrt{(u_0)^2 - (u_1')^2} \quad \Rightarrow L = \pm m \sqrt{u_\mu u^\mu}$
Example: Lagrange function for quadratic SME dispersion relations

General form is:

\[(p + k)Q(p + k) = \mu^2\]

- \(k_\nu\) is constant
- \(\mu\) is mass-like constant
- \(Q^{\mu\nu}\) is symmetric, constant; tends to \(\eta^{\mu\nu}\) in conventional limit

General procedure can be used to show that:

\[L = \mp \mu\sqrt{uQ^{-1}u} + k \cdot u\]

\(L\) is solution to a quadratic, so two cases: particle, antiparticle

Kostelecký, NR, PLB 693, 443 (2010)
Example: Find $L$ for the quadratic case of $a_\nu$ and $e_\nu$

Dispersion relation:

$$0 = p(\delta - ee)p + 2(me - a)p - m^2 + a^2$$

It follows that:

$$Q_{\mu\nu} = \eta_{\mu\nu} - e^{\mu}e^{\nu} \quad k_\nu = -a_\nu - \frac{(m - e \cdot a)}{(1 - e^2)} e_\nu \quad \mu = \frac{(m - e \cdot a)}{\sqrt{1 - e^2}}$$

The Lagrange function is:

$$L = -\frac{(m - e \cdot a)}{\sqrt{1 - e^2}} \sqrt{u^2 + \frac{(e \cdot u)^2}{1 - e^2}} - a \cdot u + \frac{(m - e \cdot a)}{1 - e^2} e \cdot u$$

Notes:

(i) When $e_\nu$ present, $a_\nu$ appears as more than a shift in $L$
(ii) homogeneity in $u$ of degree one
(iii) SME coefficients do not appear in separate terms; they are mixed
(iv) $(a_{\text{eff}})_\nu \equiv a_\nu - me_\nu$ (Kostelecký, Tasson arXiv:1006.4106)

Kostelecký, NR, PLB 693, 443 (2010)
Example: Find $L$ for the quartic case of $b_\nu$.

Quartic dispersion relation:

$$0 = (-p^2 + b^2 + m^2)^2 - 4(b \cdot p)^2 + 4b^2 p^2$$

Here, general procedure leads to a polynomial of degree 8 in $L$:

$$0 = (-b^2 (b \cdot u)^2 + b^2 L^2 - m^2 (b \cdot u)^2)^2 \times \left( b^2 u^2 - (b \cdot u)^2 + (L + m\sqrt{u^2})^2 \right) \left( b^2 u^2 - (b \cdot u)^2 + (L - m\sqrt{u^2})^2 \right)$$

Last two factors: acceptable Lagrange function:

$$L = (\mp) m\sqrt{u^2} \pm \sqrt{(b \cdot u)^2 - b^2 u^2}$$

Four solutions: analogy with particle, antiparticle, ‘spin up’, ‘spin down’

Kostelecký, NR, PLB 693, 443 (2010)
Some canonical-momentum properties

\[ p_\nu = \frac{m u_\nu}{\sqrt{u^2}} \pm \frac{(b \cdot u)b_\nu - b^2 u_\nu}{\sqrt{(b \cdot u)^2 - b^2 u^2}} \]

Solution: \( p_\mu = \text{constant}, \ u^{\mu}(\lambda) = \text{constant} \)

Notes:
\( p_\nu \) and \( u^{\nu} \) are not collinear:
Can have \( p = 0 \) and \( u \neq 0 \):
Can have \( u = 0 \) and \( p \neq 0 \)
d space

For case of \[ \mathcal{Y} \equiv \frac{1}{4} d_{\mu \nu} \tilde{d}^{\mu \nu} = 0 \]

\[ X \equiv \frac{1}{4} d_{\mu \nu} d^{\mu \nu} \]

Colladay and McDonald have found:

\[
L_d = -\frac{m}{1-2X} \left[ \sqrt{(1-2X)u^2 + ud_x^2u} \pm \sqrt{ud_x^2u} \right]
\]

Method

Disp rel \( \rightarrow \) solve for \( p_0(p) \) if factorizable \( \rightarrow \) Hamiltonian

\( \rightarrow \) Legendre transformation to get \( L(u) \)
The H$_{\mu\nu}$ coefficient gives a quartic dispersion relation

$$(\rho^2 - m^2 + 2x)^2 - 8x\rho^2 - 4\rho H\bar{H}\rho + 4y^2 = 0$$

where $x = \frac{1}{4} H_{\mu\nu} H^{\mu\nu}$ and $y = \frac{1}{4} H_{\mu\nu} \bar{H}^{\mu\nu}$

Case of $y = 0$:

$$L = -m\sqrt{u^2} \pm \sqrt{u H\bar{H}u + 2x u^2}$$

Case of $x = 0$, $y = 0$:

$$L = -m\sqrt{u^2} \pm \sqrt{u H\bar{H}u}$$

General case of $y \neq 0$ involves solving quartic for $L$. 
Having seen several Minkowski-space Lagrangians, we ask if these can be promoted to Finsler structures. Let's take the Riemann case as a start.

\[
\begin{align*}
\text{Randen} & \quad F = \sqrt{y^2} + \alpha \cdot y \\
\text{b-space} & \quad F = \sqrt{y^2} + \sqrt{b^2 y^2 - (b \cdot y)^2}
\end{align*}
\]

Is b-space a Finsler space? (Yes) Of Randen type? (No)

- Need to verify the properties of the definition.
Geometry

Consider a background vector field $b_k(x)$ in Riemann space. At any point, the velocity vector makes an angle $\theta$ with $b_k(x)$.

\[
\cos \theta = \frac{b_k y^k}{\|b \| \|y\|}
\]

We can obtain parallel and perpendicular projections of $y^i$ wrt. $b^k$:

\[
y^i_{\|} = \frac{1}{b^2} (b \cdot y) b^k
\]

\[
y^i_{\perp} = \frac{1}{b^2} \left[ b^2 y^k - (b \cdot y) b^k \right]
\]

Properties: $y_{\|} + y_{\perp} = y$

\[
y_{\|} \cdot y_{\perp} = 0
\]

(Euclidean geometry)
1. Show that $\|a\|\|y_{\parallel}\| = \pm a \cdot y^j$

From previously, we have $y_{\parallel}^j = \frac{1}{a^2} (a \cdot y) a^j$

$\Rightarrow y_{\parallel}^2 = \frac{1}{a^4} (a \cdot y)^2 a^2 = \frac{(a \cdot y)^2}{a^2}$

$\Rightarrow a^2 y_{\parallel}^2 = (a \cdot y)^2$

Take square root: $\|a\|\|y_{\parallel}\| = \sqrt{(a \cdot y)^2} = \pm a \cdot y$

$\begin{cases} + \text{ if } 0 \leq \theta \leq 90^\circ \\ - \text{ if } 90^\circ < \theta \leq 180^\circ \end{cases}$

2. Show that $\|b\|\|y_{\perp}\| = \sqrt{b^2 y^2 - (b \cdot y)^2}$

$y_{\perp}^2 = \frac{1}{(b \cdot y)^2} [b^2 y^j - (b \cdot y) b^j][b^2 y_j - (b \cdot y) b^j]$

$= \frac{1}{b^4} \left[ b^4 y^2 - 2b^2 (b \cdot y)^2 + (b \cdot y)^2 b^2 \right] = \frac{1}{b^4} \left[ b^4 y^2 - b^2 (b \cdot y)^2 \right]$

$= \frac{1}{b^2} \left[ b^2 y^2 - (b \cdot y)^2 \right]$  Result follows by taking $\sqrt{\cdot}$.
Two Finsler structures $F$ follow from the triangle properties.

\[ \|y\| \leq \|y\| \]

\[ \|y\| \leq \|y\| \]

Restrict to $\|b\| < 1$ and $\|y\| > 0$

- **Parallel:**
  \[ \|y\| > \|b\| \|y\| \|
  \Rightarrow \|y\| \pm \|b\| \|y\| > 0
  \text{i.e.} \sqrt{y^2} \pm b \cdot y > 0
  \]

- **Perpendicular:**
  \[ \|y\| > \|b\| \|y\| \|
  \Rightarrow \|y\| \pm \|b\| \|y\| > 0
  \sqrt{y^2} \pm \sqrt{b^2 y^2 - (b \cdot y)^2} > 0
  \text{(exercise)}
  \]

\[ F_a \equiv \sqrt{y^2} \pm a \cdot y \]

\[ F_b \equiv \sqrt{y^2} \pm \sqrt{b^2 y^2 - (b \cdot y)^2} \]

Randers space (use $a$) 

$b$-space
Geodesics in Finsler space

extremize arc length

$$\int ds = \int F(x, y) d\lambda$$

Geodesic equation...

$$F \frac{d}{d\lambda} \left( \frac{1}{F} \frac{dx^j}{d\lambda} \right) + \Gamma^j_i = 0$$

spray coeff.

Geodesic equation, for arc length parameter choice

$$\ddot{x}^j + \tilde{\gamma}^j_{kl} y^k y^l = \left\{ \tilde{D}(\text{SME background}) \right\}^j$$

holds for $a, b, H, (\text{all}....?)$ SME spaces

If SME background is $r$-parallel eg. $\tilde{D}_j H_{kl} = 0$,
trajectory satisfies conventional geodesic equation

$\rightarrow$ Can $r$-parallel SME backgrounds be removed by field redefinitions?
(AK, PLB 2011)
Torsions

Finsler spaces

\[ C_{jkl} := \frac{1}{2} \frac{\partial}{\partial y^i} (g_{jk}) \]

Diecke's theorem

\[ = 0 \]

\[ \neq 0 \]

Riemann space

Cartan torsion

Matsumoto torsion

Matsumoto-Hōjō theorem

\[ M_{jkl} \]

\[ = 0 \]

\[ \neq 0 \]

Randers space

Your name (?) torsion

Your name (?) theorem

\[ (?,?)_{jkl} \]

H, other SME spaces

b space ??
**Curvatures**

**Berwald h-v curvature**

\[
B P^j_{k,l m} := -\frac{1}{2} F \frac{\partial}{\partial y^k} \frac{\partial}{\partial y^l} \frac{\partial}{\partial y^m} (G^{ij})
\]

For \(a, b, H\) space, (and other SME spaces?)

- \(r\)-parallel background \(\Rightarrow G^{ij} = \tilde{\gamma}^{ij}_{kl} y^k y^l\) quadratic in \(y\)
- \(\Rightarrow B P^j_{k,l m} = 0\) ‘Berwald space’

**Converse holds for Randers space:**

- \(a\)-space with zero Berwald curvature \(\iff a(x)\) is \(r\)-parallel

**Open question:**

- SME space with zero Berwald curvature \(\iff\) SME background is \(r\)-parallel
Finally

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- Finsler structures \( \sim \) Lagrange Functions
- allow for symmetry breaking
- Structures are physically motivated by dispersion relation from Minkowski-space fermions
- Complementarity property

\[ a_\mu \leftrightarrow b_\mu \]
\[ H_{\mu \nu} \leftrightarrow H_{\mu \nu}^{-1} \] (in progress)