This is the third in a series of three lectures that were given at the IUCSS summer school during June of 2015. The lectures contain a brief introduction to the minimal Standard Model Extension (SME), an overview of renormalization in this framework, and the connection to Finsler geometry. These lectures are part of a coherent week-long program at IUCSS and therefore may rely on notation and concepts covered in other lectures.
III. FINSLER GEOMETRY AND THE SME

Scene: Enterprise is arriving at a new planet investigating ruins of an ancient civilization, they enter into a stable orbit.

Kirk: Chekov, turn around, I want another look at that ancient city down there that we just passed.

Chekov: (Makes standard turning maneuver...) I can’t believe it captain, I am going the same exact speed but we are dropping out of orbit!

Kirk: Scotty, I need those engines pronto!

Scotty: Aye captain, I’m giving you all she’s got!

Scene: Enterprise recovers orbital height, but is going much faster now, opposite to the original orbital direction.

Kirk: Spock, what happened?

Spock: Apparently captain we have just entered a region known as ”Finsler space”, only a theory until now. It was predicted to occur when Lorentz symmetry was explicitly broken in violation of the rules of Riemannian geometry in the early 21st century by a professor at the IUCSS by the name of Alan Kostelecky.

Kirk: Kostelecky, you say...

Spock: Yes captain, apparently the ancient aliens who once inhabited this world created a vector field around their planet without first considering the requirements imposed by consistency with the geometry of our universe. The local spacetime seems to have collapsed into a region known as Finsler space in which the spacetime interval depends on our velocity. Technically, the space is actually known as pseudo-Finsler due to the lack of strict positivity of...

Kirk: That’s enough Spock, get us out of here quick Scotty, the computers are not designed to calculate trajectories in this crazy space!

Attempts to construct classical models for Lorentz violation have led to the postulate that perhaps a new type of geometry called Finsler geometry may be intimately connected with the SME. Some new type of geometry is suggested by a fundamental incompatibility that has been found between Riemann-Cartan spacetime formulation
of general relativity and explicit Lorentz violation [1]. This conflict can be observed easily in the Riemann limit where the Einstein equations are given by

\[ G_{\mu\nu} = \kappa T^e_{\mu\nu}, \]  

(1)

where \( G_{\mu\nu} \) is the Einstein tensor and \( T^e_{\mu\nu} \) is the energy-momentum tensor found by varying the action with respect to the metric. The Bianchi identity is forced by the structure of the geometric theory and implies that the covariant derivative of the Einstein tensor must vanish. On the other hand, an explicit calculation yields

\[ D_\mu T^e_{\mu\nu} = J^x D^\nu k_x, \]  

(2)

where \( k_x \) is a Lorentz-violating coefficient and \( J^x \) is the associated current. For example, the \( b \)-term splits into \( k_x = b_a(x) \) as \( J^x = -\overline{\psi} \gamma^5 \gamma^a \psi \). In curved spacetime, it is generally incompatible to impose \( D^\nu b_a(x) = 0 \) as is required by the Bianchi Identity. In fact, the nonexistence of covariantly constant, nontrivial vector fields is linked to the very definition of curvature. For example, if you take a vector on a sphere and start parallel transporting it around, it is immediately clear that it can come back to where it started and have a different orientation. More explicitly, consider a vector field that satisfies \( D_\mu v^\nu = 0 \). The Riemann tensor acts on vectors according to its definition as the commutator of two covariant differentiations as

\[ [D_\mu, D^\nu] v^\rho = R^{\rho\sigma\mu\nu} v_\sigma. \]  

(3)

Remarkably, it turns out that theories involving spontaneous violation of Lorentz symmetry can evade this conflict between the geometry and the equations of motion since the entire theory prior to Lorentz breaking is fully consistent. The conflict only seems to appear when various modes of the fields in the parent model are discarded. Working with this full theory is cumbersome, so an alternative is to attempt to modify the geometry directly by incorporating the explicit background fields as forms that can influence the distance measure on the space. Such a geometry is found in Finsler geometry, a generalization of Riemann geometry in which the length measure can depend on velocities (elements of the tangent space on a manifold) as well as the locations of points in the manifold.
To see the connection between the SME and Finsler geometry, look at the classical
lagrangian derived in the lecture by Neil Russell in a previous lecture given by [2]

\[ L_{ab}^\pm = -m\sqrt{u^2} - a \cdot u \mp \sqrt{(b \cdot u)^2 - b^2 u^2}, \quad (4) \]
derived for the case of constant \( a \) and \( b \) vector fields on a Minkowski spacetime
background. In this expression, \( u^\mu = dx^\mu/d\lambda \) is the particle four-velocity. It is
possible to generalize the above lagrangian by introducing a Riemann metric \( r_{\mu\nu}(x) \)
and allowing the constant backgrounds to become one-forms that can also depend on
\( x \). It is likely that a Foldy-Wouthuysen transformation can be used to justify this
generalization by using an appropriate approximation, but the details remain to be
fully worked out.

Technically, Finsler geometry requires a positive definite metric, but the above
Lagrangian contains a metric with negative signature. It is therefore convenient to
re-express the Lagrangian by using a Wick rotation in terms of a Euclidean-signature
metric. Care must be used in employing this transformation as part of the resulting
Finsler space may be inaccessible to the physical particle due to the loss of causal
limitations on the speed. The result is a function that strongly resembles a Finsler
function

\[ F_{ab} = \sqrt{y^2 + a \cdot y \pm \sqrt{b^2 y^2 - (b \cdot y)^2}}, \quad (5) \]
where \( y^i = dx^i/dt \), with \( i = 1, 2, \ldots, n \) label the velocity components after Wick
rotation. The problem is then to find a curve that extremizes the length functional

\[ \int_A^B F(x, y)d\lambda, \quad (6) \]
where \( \lambda \) is some parametrization of the underlying curve.

An important property of this function is that it is homogeneous of degree 1. A
homogeneous function of degree \( r \) is defined by the condition \( H(\lambda y) = \lambda^r H(y) \). It
satisfies the important relation \( y^i (dH/dy^i) = rH \) as can be shown by differentiation
of the previous relation with respect to \( \lambda \) and setting \( \lambda \to 1 \) afterwards. This fact
can be used to demonstrate \( g_{ij} y^i y^j = F^2 \), provided the Finsler metric is defined as

\[ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}. \quad (7) \]
Examination of various limits of $F_{ab}$ gives good insight into some possibilities.

As a first case, consider $a = b = 0$. In this case, $g^{ij} = r^{ij}(x)$ is simply the conventional Riemann metric. The case $b = 0$ gives the Finsler metric a velocity dependence as

$$g_{ij} = \frac{F}{\sqrt{y^2}} \left( r_{ij} - \tilde{l}_i \tilde{l}_j \right) + (\tilde{l}_i + a_i)(\tilde{l}_j + a_j),$$  

(8)

where $\tilde{l}_i = r_{ij}y^j / \sqrt{y^2}$. This case is known as Randers space and was investigated by Randers [4] as an attempt at including electromagnetism into the geometry of space-time. The metric can be used to generate a geodesic equation that is the same form as that of general relativity, but the Christoffel symbols gain a velocity dependence.

The case of $a = 0$ corresponds to a more complicated case involving spin couplings. It is natural to try to use $F_-$ and $F_+$ for two different Finsler functions, however, either one taken independently has a metric that becomes singular where $y^i$ is parallel to $b$. In addition, $F_-$ fails to be convex in a finite region around this singular point. Examination of the indicatrix $F = 1$ suggests combining the two Finsler functions into a single algebraic variety

$$f(F, y_p, y^i_p) = (F^2 - y^2)^2 - b^2 y_p^2 (2(F^2 + y^2) - b^2 y_p^2) = 0,$$  

(9)

where $y_i$ is split into components parallel and perpendicular to $b^i$. On the indicatrix, the singular point is a cone singularity which can be desingularized using a standard blowup procedure. Once this is accomplished, a new type of geometry is produced that involves defining a Finisher function of the lifted manifold that sits above the algebraic variety [5]. The only issue that remains is that $det(g) = 0$ on a ring where the magnitude of the velocity is of order $b$. The details of this have been worked out and will appear shortly.

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**REFERENCES**

5. D. Colladay and P. McDonald, To appear shortly.