Prelude: Outline

Outline:
1. Title page.
2. Prelude: Very rough ideas of noncommutative geometry.
3. General Theory: Turning regular theories into NC theories. NCQED.
4. Immediate consequences of noting NCQED ⊂ SME.
5. Experimental tests: Lamb shift, clock-comparisons, etc.

Finally:
Thank you to the IUCSS for inviting me!
Imagine that there is a Heisenberg uncertainty relation for position coordinates:

\[ \Delta x \Delta y > 0 \]

This corresponds to noncommutativity between position coordinates:

\[ [x, y] > 0 \]

We can make this compatible with observer Lorentz symmetry:

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu} \]

where \( \theta^{\mu\nu} \) is real, antisymmetric, and spacetime-constant.

Prelude: Very Rough Idea of Noncommutative Geometry
(from a math perspective)

Relatively old mathematical idea:
Commutative algebras of functions $\leftrightarrow$ Geometric spaces

Relatively new mathematical idea:
Noncommutative algebras of functions $\leftrightarrow$ Other types of spaces?
(E.g., matrix-valued functions)

Noncommutative geometry is a very large subject. I have only explored a small part of it.

My goal in this talk is only to give you a glimpse of it.
Consider an ordinary theory with functions/fields $f$, $g$, ...
This may be turned into a noncommutative field theory with $\hat{f}$, $\hat{g}$,... by replacing all ordinary products with \textbf{Moyal $\star$ products}:

$$(f \cdot g)(x) \rightarrow (\hat{f} \star \hat{g})(x) := \exp \left( \frac{1}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) \hat{f}(x)\hat{g}(y) \bigg|_{x=y}$$

$$= \hat{f}(x)\hat{g}(x) + \frac{1}{2} \theta^{\mu\nu} \frac{\partial \hat{f}}{\partial x^\mu} \frac{\partial \hat{g}}{\partial x^\nu} + o(\theta)^2$$

Notes:

1. $\hat{a} \star \hat{f} = a \hat{f}$ for any constant $a$.
2. $[x^\mu, x^\nu] \rightarrow [\hat{x}^\mu, \hat{x}^\nu]_\star = i\theta^{\mu\nu}$ as desired.
3. $\theta^{\mu\nu}$ is antisymmetric.
4. This is sort of like a Taylor series $\Rightarrow$ nonlocality.

\( \mathcal{L} = \frac{1}{2} i \hat{\psi} \star \gamma^\mu \hat{D}_\mu \hat{\psi} - m \hat{\psi} \star \hat{\psi} - \frac{1}{4} \hat{F}_{\mu \nu} \star \hat{F}^{\mu \nu} \)

where

\[ \hat{F}_{\mu \nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \left[ \hat{A}_\mu, \hat{A}_\nu \right] \]

and

\[ \hat{D}_\mu \hat{\psi} := \partial_\mu \hat{\psi} - i q \hat{A}_\mu \star \hat{\psi} \]

Warning! \( \hat{\psi} \) does not necessarily correspond to a physical fermion!

1. It is a noncommutative field.

2. It follows a nonconventional gauge transformation.

Similarly, \( \hat{A}_\mu \) does not necessarily correspond to a physical photon!

\( \Rightarrow \) Interpretation is nontrivial!
The Seiberg-Witten map $\hat{\psi}, \hat{A}_\mu \rightarrow \psi, A_\mu$ eases interpretation:

1. $\psi, A_\mu$ are ordinary fields.
2. They follow ordinary gauge transformations.
3. Their physical behavior is physically equivalent to $\hat{\psi}, \hat{A}_\mu$

Explicitly for NCQED,

$$\hat{\psi} = \psi - \frac{1}{2} \theta^{\alpha\beta} A_\alpha \partial_\beta \psi + o(\theta)^2$$

$$\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + o(\theta)^2$$

N. Seiberg and E. Witten, JHEP 09 (1999) 032.
A.A. Bichl et al., TUW-01-03, UWTHPH-2001-9 (2001); hep-th/0102103.
General Theory: Fermion/Photon Action

\[ \mathcal{L} = \frac{1}{2} i \bar{\psi} \gamma^\mu \overset{\rightarrow}{D}_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} (F \cdot F) \]

\[ - \frac{1}{8} i q (\theta \cdot F) \bar{\psi} \gamma^\mu \overset{\rightarrow}{D}_\mu \psi + \frac{1}{4} i q \theta^{\alpha \beta} F_{\alpha \mu} \bar{\psi} \gamma^\mu \overset{\rightarrow}{D}_\beta \psi \]

\[ + \frac{1}{4} m q (\theta \cdot F) \bar{\psi} \psi \]

\[ - \frac{1}{2} q \theta^{\alpha \beta} F_{\alpha \mu} F_{\beta \nu} F^{\mu \nu} + \frac{1}{8} q (\theta \cdot F) (F \cdot F) \]

**Conventional QED**

**New fermion-photon interactions**

**Modified photon dynamics**

**Initial observations:**
1. This is manifestly U(1) gauge invariant.
2. The new terms break particle Lorentz symmetry.
3. The new interactions have mass dimension >4.

⇒ Noncomm QED is contained in the (nonminimal) Standard-Model Extension!

Immediate Consequences: Free fermions, Neutral Fermions

\[ \mathcal{L} = \frac{1}{2} i \bar{\psi} \gamma^\mu \overset{\leftrightarrow}{D}_\mu \psi - m \bar{\psi} \psi - \frac{1}{4} (F \cdot F) \]
\[ - \frac{1}{8} i q (\theta \cdot F) \bar{\psi} \gamma^\mu \overset{\leftrightarrow}{D}_\mu \psi + \frac{1}{4} i q \theta^{\alpha\beta} F_{\alpha\mu} \bar{\psi} \gamma^\mu \overset{\leftrightarrow}{D}_\beta \psi \]
\[ + \frac{1}{4} m q (\theta \cdot F) \bar{\psi} \psi \]
\[ - \frac{1}{2} q \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} + \frac{1}{8} q (\theta \cdot F) (F \cdot F) \]

Fermion observations:
1. Free fermions propagate conventionally.
2. Neutral fermions are unaffected.
Photon observations:
1. Even in vacuum, electrodynamics is nonlinear.
2. While this action is part of the nonminimal SME, the new photon terms have never been explicitly studied. (Existing studies concentrate on quadratic terms.)
Other observations:
1. As long as $\theta$ is spacetime-constant, energy and momentum are conserved.
2. The full combination CPT is preserved.
3. All other combinations of C, P, and T are broken.
4. Since there are no new operators of mass dimension 3 or 4, the fermionic sector has conventional stability and causality.
5. The conventional spin-statistics connection holds.


Finally: All of these observations apply to any noncommutative theory that reduces to the Standard Model, not just NCQED.
If $\theta^0 j = 0$, then only conventional time derivatives appear
$\Rightarrow \psi$ propagates with unitary time evolution.

If $\theta^0 j \neq 0$, then nonconventional time derivatives appear
$\Rightarrow \psi$ may propagate with nonunitary time evolution.

1. However, if $\theta_{\mu \nu} \theta^{\mu \nu} \geq 0$ and $\epsilon_{\mu \nu \alpha \beta} \theta^{\mu \nu} \theta^{\alpha \beta} = 0$, then there exists an inertial frame with $\theta^0 j = 0 \Rightarrow \psi$ propagates with unitary time evolution.

2. Alternately, a field redefinition $\psi = A \chi$ may lead to a physical fermion field $\chi$ that does propagate with unitary time evolution.
Common experimental situation: $F \rightarrow f + F$, where
1. $f$ is a spacetime-constant background field
2. This new $F$ is a small dynamical fluctuation.

Then the NCQED lagrangian is equivalent to the mSME lagrangian with:

$$q_{\text{eff}} = (1 + \frac{1}{4} q \theta \cdot f) q$$

$$c_{\mu\nu} = -\frac{1}{2} q f_{\mu}^{\lambda} \theta_{\lambda\nu}$$

$$k_{F\alpha\beta\gamma\delta} = (-q f_{\alpha}^{\lambda} \theta_{\lambda\gamma} \eta_{\beta\delta} + \frac{1}{2} q f_{\alpha\gamma} \theta_{\beta\delta} - \frac{1}{4} q f_{\alpha\beta} \theta_{\gamma\delta})$$

$$-(\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + (\alpha \beta \leftrightarrow \gamma\delta)$$

Note: $q_{\text{eff}}$, $c_{\mu\nu}$, and $k_{F\alpha\beta\gamma\delta}$ are proportional to the background EM field.
The potential experienced by an electron in a hydrogen-like atom receives an extra coupling to angular momentum:

\[
H \supset \frac{p^2}{2m} - \frac{Z\alpha}{r} - \frac{Z\alpha}{4r^3} \epsilon_{jkl} L^j \theta^{kl} + o(\theta)^2
\]

Precise agreement between experimental measurement and conventional theory indicates that this last term must be small:

\[
|\theta^{kl}| \lesssim (20 \text{ GeV})^{-2}
\]

Expt Tests: Overview of Clock-comparison experiments

a. How to build an atomic clock

Atom in uniform $\mathbf{B}$-field, with ladder of energy levels

Thursday 11:00-12:30: Arnaldo Vargas, Much, Much More about Clock-Comparison Experiments
Expt Tests: Overview of Clock-comparison expts

b. How to read an atomic clock

Important points

1. Frequency measurements can probe atomic energy levels
2. $\vec{B}$ field defines orientation of clock: "clock axis"
3. In conventional physics, clock frequency is independent of clock axis and clock velocity.
Expt Tests: Overview of Clock-comparison expts

c. How clocks are connected to the SME

Suppose the frequency of a given clock depends on its orientation and/or velocity
⇒ Rotation and/or boost violation
⇒ Lorentz violation

\[ \approx b_\mu \]
Expt Tests: Overview of Clock-comparison expts

c. How clocks are connected to the SME

Frame-Dependent Energy-Level Shifts in atomic clocks!

- Theory predicts them!
- Experiment seeks them!

Typical sensitivity in atomic expts:

\[ \text{Lorentz-violation coefficients} \lesssim 10^{-23} \text{ to } 10^{-33} \text{ (GeV)} \]
Expt Tests: Time Variation in Earth-Based Expts

Convenient Coordinate Systems

Quantization axis of experiment

\( \hat{X}, \hat{Y}, \hat{Z} = \text{nonrotating frame, origin at Sun's center} \)

\( \hat{x}, \hat{y}, \hat{z} = \text{rotating frame, origin in laboratory} \)
If the EM field is a magnetic field along $z$, then this term creates a quadrupole-type energy shift to certain atomic states $|F, m_F\rangle$.

Recall that, in the presence of a constant external EM field, noncommutativity induces an effective $c_{\mu\nu}$ term:

$$c_{\mu\nu} = -\frac{1}{2} q f_{\mu}^{\lambda} \theta_{\lambda\nu}$$
Expt Tests: Atomic Clocks in External Magnetic Fields

As Earth rotates, $x$ and $y$ axes rotate
$\Rightarrow$ component $\theta_{xy}$ varies at Earth’s sidereal rotation frequency $\Omega$:

$$\theta_{xy} = \theta_{YZ} \sin \chi \cos \Omega t + \theta_{ZX} \sin \chi \sin \Omega t$$

$\Rightarrow$ Energy shift

$$\delta E = E_0 + \left( \overline{m_F B \sin \chi \sum_w q_w m_w \gamma_w} \right) (\theta_{YZ} \cos \Omega t + \theta_{ZX} \sin \Omega t)$$

but **Sidereal** Time Dependence

**Quadrupole** Spectrum Shift, but **Sidereal** Time Dependence

Gap between adjacent energy levels

1 sidereal day

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<td>$-3/2$</td>
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Expt Tests: Atomic Clocks in External Magnetic Fields

Test 1: $^{199}\text{Hg}/^{133}\text{Cs}$ (Berglund et al., PRL 75 (1995) 1879)

Electrons have $J \leq 1/2$, so $\gamma_e = 0$.

$^{133}\text{Cs}$ nucleus has $I = 7/2$ with a valence proton, so $\gamma_p \sim 10^{-3}$.

Magnetic field $B \sim 5$ mG, bound on sidereal variations $\sim 100$ nHz.

$$|\theta_{YZ}|, |\theta_{ZX}| \lesssim \left( \frac{1}{10} \text{ GeV} \right)^{-2}$$

Weak bound is due to small magnetic field.

Test 2: $^{9}\text{Be}^+/^{1}\text{H}$ (Prestage et al., PRL 54 (1985) 2387)

Electrons have $J \leq 1/2$, so $\gamma_e = 0$.

$^{9}\text{Be}$ nucleus has $I = 3/2$ with a valence neutron, so $\gamma_p \sim 0 \Rightarrow$ No sensitivity?

More careful nuclear model gives $\gamma_p \sim 10^{-4}$.

Magnetic field $B \sim 1$ T, bound on sidereal variations $\sim 100$ $\mu$Hz.

Best bound from atomic clocks in external magnetic fields:

$$|\theta_{YZ}|, |\theta_{ZX}| \lesssim (10 \text{ TeV})^{-2}$$
Expt Tests: Bound using nuclear gluon field?

Remember that atomic Coulomb potential 
\[-\frac{Z\alpha}{r} \rightarrow -\frac{Z\alpha}{r} - \frac{Z\alpha}{4r^3} \varepsilon_{jkl} L^j \theta^{kl}\]

Guess: Maybe similar modifications occur to nuclear gluonic field.
More precise guess: Suppose that
\[V_{\text{Coulomb}} \supset \kappa (\vec{\theta}_B \cdot \vec{L}) \rightarrow \kappa (\vec{\theta}_B \cdot \vec{J}) \rightarrow \frac{d\theta}{2} \theta_{\mu\nu} \bar{N} \sigma_{\mu\nu} N \rightarrow d\theta (\vec{\theta}_B \cdot \frac{\vec{S}}{S})\]

Based on nuclear theory, estimate \(d\theta \sim 0.1 \text{ GeV}^3\)

Comparison to \(^{199}\text{Hg}/^{133}\text{Cs}\) clock-comparison experiment then gives
\[|\theta| \lesssim (5 \times 10^{14} \text{ GeV})^{-2} \leftarrow \text{Extremely tight bound!}\]

Partial support: Radiative corrections in NCQCD indeed induce an interaction

\[ \mathcal{L} \supset \frac{\alpha_s}{12\pi} m \Lambda^2 \theta^{\mu\nu} \overline{N} \sigma_{\mu\nu} N \]

where \( \Lambda \) = ultraviolet regularization scale.

With this, the \(^{199}\text{Hg}/^{133}\text{Cs}\) clock test gives \( \theta \Lambda^2 \lesssim 10^{-29} \)


Expt Tests: Other.

Other experimental tests arise from high-energy collisions, electron EDMs, and astrophysical photons. The bounds from these are typically weaker than those from low-energy atomic tests.
1. Key physical motivation is allowing $[x, y] > 0$.
2. Any realistic noncommutative theory is part of the SME.
3. Strong arguments imply the NC scale is $> 10$ TeV.
   Plausible arguments imply the NC scale is $> 10^{14}$ GeV.