Notes on the classical-particle limit of the SME

Presentation at the...

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1. Introduction and Motivation

Problem at hand is to find the trajectory of a classical particle in spacetime, when Lorentz violation occurs.

There exists a broad variety of results for Lorentz violation in the context of field theory. We want to find the classical limit of those results.

Trajectories are determined by the variational principle. What is the action to be extremized?

Relevant ideas:
- classical particles are localized wave packets moving at the group velocity
- starting point is to look at flat Minkowski spacetime
- in variational calculus, motion is governed by a Lagrange function
- Lagrange function in curved spacetime has many of the features of Finsler geometry
- An important link between the field theory and the classical motion is the relation between energy and momentum: the dispersion relation.
Conventional Lorentz-preserving Lagrange function

Consider paths from event A to event B...

The quantity to extremize is the spacetime interval:

\[
\begin{align*}
\text{infinitesimal} & \quad d\tau^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\
\text{metric has signature} & \quad \left[+,-,-,-\right] \\
\text{we assume timelike intervals along the path} & \quad \text{so that RHS} > 0 \\
\text{Notation:} & \quad \text{on diag} \\
\text{Note:} & \quad \text{off diag}
\end{align*}
\]

\[
\begin{align*}
m_{\mu\nu}(x) & \quad \gamma_{\mu\nu} = \delta_{\mu\nu} = \left\{ \begin{array}{ll} 1 & \text{on diag} \\ 0 & \text{off diag} \end{array} \right. \\
\text{So} & \quad d\tau^2 = dx^\mu m_{\mu\nu} dx^\nu
\end{align*}
\]

iv. this signature is used for QFT literature; opposite signs common for gravity literature.

For a path from A to B, the action \( S \) is the sum of the spacetime intervals along the path, representing the passage of proper time:

\[
S = -m \int_A^B d\tau \\
(\mu = m^\alpha\alpha)
\]

Parametrizing with \( \lambda \), which might be the proper time, we have:

\[
d\tau = \sqrt{dx^\mu m_{\mu\nu} dx^\nu} = \sqrt{u^\mu m_{\mu\nu} u^\nu} d\lambda
\]
\[ d\tau = \sqrt{dx^\alpha r_{\alpha\nu} dx^\nu} = \sqrt{u^\mu r_{\mu\nu} u^\nu} \, d\lambda \]

with \( u^\mu \equiv \frac{dx^\mu}{d\lambda} \), \( u \)-velocity.

\[ S = -m \int_A^B \sqrt{u\dot{u}} \, d\lambda \quad r = r(x) \]

Euler-Lagrange equation gives the equations of motion in the manifold, and the result is the 'geodesic equation':

\[ \dot{u}^\mu = \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \]

Christoffel symbols \( \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left( \partial_\nu r_{\alpha\beta} - \partial_\alpha r_{\nu\beta} - \partial_\beta r_{\nu\alpha} \right) \)

The Lagrange function is the argument of the action integral:

\[ S = \int_A^B L(x, u, \lambda) \, d\lambda \]

\[ \Rightarrow \text{ Conventional particle } \quad L(x, u, \lambda) = -m \sqrt{u\dot{u}} \]
Finsler geometry (almost)

\[ L = -m \sqrt{u \cdot u} \quad r = r(x) \]

Consider \( (\frac{L}{m})^2 = u \cdot u \)

\[ \frac{\partial}{\partial u^m} \left( \frac{L}{m} \right)^2 = (ru)_m + (ur)_m = 2(ru)_m \]

\[ \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial u^\beta} \left( \frac{L}{m} \right)^2 = 2 \, g_{\mu\nu} \]

\[ \Rightarrow \quad g_{\mu\nu}(x) = \frac{1}{2m} \frac{\partial}{\partial u^\gamma} \frac{\partial}{\partial u^\rho} (L^2) \]

The metric can be recovered from the Lagrange function.

Conventional metric is space-time isotropic

If the Lagrange function contains Lorentz violation, then we'd get a metric containing Lorentz violation embedded in the way we measure on the manifold.

\[ g_{\mu\nu}(x, \phi, \theta, \ldots) = \frac{1}{2m} \frac{\partial}{\partial u^\rho} \frac{\partial}{\partial u^\sigma} L^2 \]

\[ r \Rightarrow (t + + \ldots) \quad \text{Riemann} \]

\[ r \Rightarrow (t, \ldots \ldots \ldots) \quad \text{Pseudo Riemann} \]
Minkowski limit

Difficult to handle curved space time and Lorentz violation, so ...

... restrict attention to the flat spacetime limit:

- metric is constant: \( r \rightarrow \eta_{\mu\nu} \)

\[
\eta_{\mu\nu} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1
\end{pmatrix}
\]

- Conventional Lagrange function is

\[
L = -m \sqrt{u^\mu u^{\nu}}
\]

\( u^\mu \equiv u^\mu \eta_{\mu\nu} = u^\nu u_\nu \)

Allow Lorentz-breaking background fields on this Minkowski spacetime.

SME background fields \( a_\nu, b_\nu, \ldots \) are also constant in this flat space.

\[ \Rightarrow \text{No position dependence: } \]

\[
L = L(u, \lambda) \quad \frac{\partial L}{\partial x^\mu} = 0
\]

We suppose that the physics does not depend on the path parameter \( \lambda \):

\[
L = L(u) \quad \frac{\partial L}{\partial \lambda} = 0
\]
Homogeneity of the Lagrange function

We have \( L = L(u) \), where

\[
u^\mu = \frac{dx^\mu}{d\lambda}\]

The physics should be unchanged under any path reparametrization.

This holds if \( L \) is homogeneous of degree one in the 4-velocity:

\[
\text{For example: } \quad d\tau = \sqrt{\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \ d\lambda = \sqrt{\frac{dx^\mu}{d\alpha} \frac{dx^\nu}{d\alpha}} \ d\alpha
\]

\[ \text{or } \alpha \]

Reminders: **homogeneity.**

\[
\begin{array}{cc}
\text{Function} & \text{degree of homogeneity} \\
\hline
f(x, y) = x + y & 1 \\
f(x, y) = x^2 + y^2 & 2 \\
f(x, y) = \sqrt{x^2 + y^2} & 1 \\
f(x, y) = \sqrt{x - y} & \frac{1}{2} \\
\frac{\partial f}{\partial x} & n - 1, \text{ for } f(x) \text{ of degree } n
\end{array}
\]

Euler's result:

\( f \) is homogeneous of degree \( n \) in \( xd \)
Exercise: verify the degree of homogeneity for the examples above.

Using this homogeneity requirement, it follows that

\[ u^\nu \frac{\partial L}{\partial u^\nu} = L \]

\[ \Rightarrow \quad L(u) = -u^\nu p_\nu \]

Where \( p_\nu = -\frac{\partial L}{\partial u^\nu} \) is the canonical momentum.
Minimal-SME Dispersion relation

In conventional physics, where Lorentz symmetry is preserved, the conserved energy and momentum of a free particle satisfy a quadratic 'dispersion relation'

\[ p^2 = m^2 \quad \text{and} \quad (p_0)^2 - \vec{p}^2 = m^2 \]

The dispersion relation for a free fermion in flat spacetime, with minimal Lorentz violation is quartic in the 4-momentum.

\[ 0 = \frac{1}{4} \left( V^2 - S^2 - A^2 - \bar{S}^2 \right)^2 + V^2 A^2 - (V \cdot A)^2 + 4 \left[ \mathcal{P}(VTA) - S(\nabla \cdot A) - V \nabla \cdot V + A \nabla \cdot A \right] - X(V^2 + S^2 - A^2 - \bar{S}^2) - 2 \mathcal{P} + X^2 + Y^2 \]

\{' \text{all with T} \}

Where:

- **Scalar**: \( S \equiv -m + e \cdot p \)
- **Pseudoscalar**: \( \mathcal{P} \equiv f \cdot p \)
- **Vector**: \( V_\mu = p_\mu + (\vec{e} p)_\mu - a_\mu \)
- **Axial vector**: \( A_\mu \equiv (\vec{e} p)_\mu - b_\mu \)
- **Tensor**: \( \tilde{T}_{\mu \nu} \equiv \frac{i}{2} (g_{\mu \lambda} \gamma^\lambda - H_{\mu \nu}) \)

\[ \text{and} \]

\[ \tilde{T}_{\mu \nu} = \frac{1}{i} \epsilon_{\mu \nu \alpha \beta} T^{\alpha \beta} \]

\{' \text{dual tensor} \}

\[ X \equiv \tilde{T}_{\mu \nu} \tilde{T}^{\mu \nu} ; \quad Y \equiv \tilde{T}_{\mu \nu} T^{\mu \nu} \quad \text{invariants} \]

Version here is Eq. (1) in PLB 693, 443 (2010) [arXiv:1008.5062]
Particular dispersion relations:

1. Dispersion relation for $f$ and $a$:
   Set $b = 0, c = 0, d = 0, e = 0, g = 0$:

   \[
   \text{Scalar: } S \equiv -m + c \cdot p \quad S = -m
   \]

   \[
   \text{Pseudoscalar: } P \equiv f \cdot p \quad P = f \cdot p
   \]

   \[
   \text{Vector: } V_{\mu} = p_{\mu} + (c \cdot p)_{\mu} - a_{\mu} \quad V = p - a
   \]

   \[
   \text{Axial vector: } A_{\mu} \equiv (c \cdot p)_{\mu} - b_{\mu} \quad A = 0
   \]

   \[
   \text{Tensor: } T_{\mu \nu} \equiv \frac{1}{2} (g_{\mu \nu} p^2 - H_{\mu \nu}) \quad T = 0
   \]

   Using these substitutions, only the first line contributes:

   \[
   0 = \frac{1}{4} \left( V^2 - S^2 - A^2 - p^2 \right)^2
   \]

   \[
   + 4 \left[ P (V \cdot A) - S (V \cdot A^2) - V T \cdot V + A T \cdot A \right]
   \]

   \[
   + V^2 A^2 - (V \cdot A)^2
   \]

   \[
   - X (V^2 + S^2 - A^2 - p^2) - 2 V S P + X^2 + Y^2
   \]

   \[
   \Rightarrow 0 = V^2 - S^2 - p^2
   \]

   \[
   \Rightarrow 0 = (p - a)^2 - m^2 - (f \cdot p)^2
   \]

2. Dispersion relation for $b$: set all other coefficients zero
\[ S = -m \]
\[ P = 0 \]
\[ V = p \]
\[ A = -b \]
\[ T = 0 \]

\[ \begin{align*}
\mathcal{O} &= \frac{1}{4} \left( V^2 - S^2 - A^2 - P^2 \right)^2 \\
&+ 4 \left[ P(V \cdot A) - S (V \wedge A) - V T T V + A T T A \right] \\
&+ V^2 A^2 - (V \cdot A)^2 \\
&- X (V^2 + S^2 - A^2 - P^2) - 2 Y S P + X^2 + Y^2 \\
\end{align*} \]

\[ \Rightarrow \quad \mathcal{O} = \frac{1}{4} \left( V^2 - S^2 - A^2 \right)^2 + V^2 A^2 - (V \cdot A)^2 \]

\[ \mathcal{O} = \left( p^2 - m^2 - b^2 \right)^2 + 4 p^2 b^2 - 4 (p \cdot b)^2 \]

**Exercises:**
1. Write the dispersion relation for the \( H \) coefficient, using the definitions for the invariants \( X \) and \( Y \). Answer is below Eq. (14) in PLB 693, 443 (2010) [arXiv:1008.5062]
2. Write the dispersion relation for antisymmetric \( d_{\mu\nu} \) coefficient. Answer is Eq. (13) in PRD 91, 045008 (2015) [arXiv:1501.02490]
2. General method for Lagrange function in flat spacetime to match dispersion relation

We seek a Lagrange function for a classical particle that generates a dispersion relation identical to the field-theory one.

So, it's necessary for the classical velocity $v^i$ to match the group velocity $v^i_g$ found from the dispersion relation.

Mathematically, this is expressed as:

$$v^i = v^i_g$$

for $j = 1, 2, 3$.

Note that:

- **Classical**:
  $$v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{u^i}{u^0}$$

- **Quantum**:
  $$v^i_g = \frac{\partial \rho^0}{\partial p^j} = - \frac{\partial \rho^0}{\partial p^j}$$

So, three velocity conditions:

$$\frac{u^i}{u^0} = - \frac{\partial \rho^0}{\partial p^j}$$

We also require that the Lagrange function be homogeneous in the 4-velocity:

$$L = -p^\tau u^\tau$$

and that the momentum satisfies the desired dispersion relation:
and that the momentum satisfies the desired dispersion relation:

\[
\mathcal{R} \left( m, \alpha, \beta, \ldots ; \ p \right) = 0
\]

The boxed equations constitute 5 conditions on 9 variables: \( p_\nu, u^\mu, \) and \( L. \)

In principle, we can use 4 conditions to eliminate the momenta \( p_\nu \) from the remaining condition. This leaves a polynomial equation in \( L \) with coefficients that depend only on the 4-velocity \( u^\mu: \)

\[
P(L ; u) = 0
\]

The roots of this polynomial give Lagrange functions \( L(u; a, b, c, \ldots) \) which meet the above requirements.

Notes:

1. Only some roots are acceptable
2. Multiple roots reflect idea of separate trajectories for particles, antiparticles, spin up, spin down; all have the same dispersion relation.
Show that the conventional Lagrange function \( L(u) = \mp m \sqrt{u^2} \) results from this method if the dispersion relation \( p_0^2 - p_1^2 = m^2 \) is imposed.

**Example: conventional particle in 1D**

**Solution**

There are 3 conditions (in 1D):

- **Dispersion rel:** \( p_0^2 - p_1^2 = m^2 \) \[-(1)\]
- **Homogeneity:** \( L = -p_0 u^0 - p_1 u^1 \) \[-(2)\]
- **Velocity:** \( \frac{u^1}{u^0} = -\frac{\partial p_0}{\partial p_1} \) \[-(3)\]

These are imposed on 5 variables, \( L, u_0, u^1, p_0, p_1 \).

We must eliminate the two momenta...

* **Implicit differentiation of (1) wrt \( p_1 \):**

\[
2 p_0 \frac{\partial p_0}{\partial p_1} - 2 p_1 = 0 \implies \frac{\partial p_0}{\partial p_1} = \frac{p_1}{p_0}
\]

\( \implies \) eq \((3)\) becomes \( \frac{u_1}{u_0} = -\frac{p_1}{p_0} \) \[\text{\(\text{\(3')\)}\} \]

* **Use (3') to eliminate \( p_1 \) from (1):**

\[
\begin{align*}
p_0^2 - \left( -\frac{u_1}{u_0} p_0 \right)^2 &= m^2 \\
\implies p_0^2 \left[ 1 - \left( \frac{u_1}{u_0} \right)^2 \right] &= m^2 \\
\implies p_0 &= \frac{\mp m}{\sqrt{1 - \left( \frac{u_1}{u_0} \right)^2}} \tag{\(\text{\(1')\)}\}
\end{align*}
\]

* **Use (3') to eliminate \( p_1 \) from (2):**

\[
L = -p_0 u^0 - \left( -\frac{u_1}{u_0} p_0 \right) u^1
\]
\[ \Rightarrow L = -P_0 u_o \left[ 1 - \left( \frac{u_1}{u_o} \right)^2 \right] \quad -(2') \]

* Use \((1')\) to eliminate \(P_0\) from \((2')\)

\[ (2') : \quad L = -P_0 u_o \left[ 1 - \left( \frac{u_1}{u_o} \right)^2 \right] \]

\[ = \frac{m u_o}{\sqrt{1 - \left( \frac{u_1}{u_o} \right)^2}} \left[ 1 - \left( \frac{u_1}{u_o} \right)^2 \right] \]

\[ \Rightarrow L = \mp m \sqrt{(u_o)^2 - (u_1)^2} \]

\[ \Rightarrow L = \mp m \sqrt{u^2} \]
Case of $b$ coefficient

Dispersion relation is

$$\sigma = \left( p^2 - m^2 - b^2 \right)^2 - 4 \left( b \cdot p \right)^2 + 4 b^2 p^2$$

Process of eliminating the momenta is involved and software like Maple or Mathematica is useful!

Eventually, the polynomial in $L, u^\nu$ can be put into the following factored form:

$$\sigma = \left[ b^2 u^2 - (b \cdot u)^2 + \left( L + m \sqrt{u^2} \right)^2 \right] \cdot \left[ b^2 u^2 - (b \cdot u)^2 + \left( L - m \sqrt{u^2} \right)^2 \right] \cdot \left[ -b^2 (b \cdot u) + b^2 L^2 - m^2 (b \cdot u)^2 \right]^2$$

The first factor reveals the root

$$\Rightarrow L = -m \sqrt{u^2} \pm \sqrt{(b \cdot u)^2 - b^2 u^2}$$

Notes

i. Second factor reverses the sign of the first term

ii. These solutions agree with earlier discussion

iii. The polynomial here is of degree 8 in $L$. Four of the solutions are disallowed

iv. This method has been employed to find the Lagrange function for the $H_{\mu \nu}$ coefficient with $Y \neq 0$.

The solution involves solving a quartic polynomial for $L^2$. The resulting expression is long.

3. Quadratic dispersion relation:

Any quadratic dispersion relation in spacetime can be cast into the following form:

\[(p + k) \Omega \sim (p + k) = \mu^2\]

\[p_\gamma = \text{conserved momentum}\]
\[k_\gamma, \Omega^{\mu \nu}, \mu \text{ are constants}\]

In Lorentz-preserving limit,

\[\mu \rightarrow m\]
\[k_\gamma \rightarrow 0\]

\[\Omega^{\mu \nu} \rightarrow \delta^{\mu \nu}\]

\[\Rightarrow p_\gamma^2 = m^2\]

A Lagrange function yielding this dispersion relation is

\[L = -\mu \sqrt{u \Omega^{-1} u} + k \cdot u\]

Example: $a$ and $f$ coefficients

The dispersion relation for the $a$ and $f$ coefficients is:

\[(p-a)^2 - (p\cdot f)^2 - m^2 = 0 \quad \text{--- (1)}\]

→ Match with canonical form,

\[(p+k \cdot \Omega \cdot (p+k) = \mu^2, \quad \text{--- (2)}\]

to find $\Omega^M \nu$, $\kappa$, $\mu$, ...

Calculation

Multiply (1) and (2) out, match coefficients:

(1) $\Rightarrow$ $p^2 - 2a \cdot p + a^2 - (p\cdot f)(f\cdot p) - m^2 = 0$

$\Rightarrow$ $p (\delta - ff)p - 2a \cdot p - m^2 + a^2 = 0$

(2) $\Rightarrow$ $p \Omega p + 2k \Omega p + k \Omega k - \mu^2 = 0$

\[
\begin{cases} 
\Omega^M \nu = \delta^M \nu - f^M f^\nu & p^2 \text{ coeff.} \quad \text{--- (3)} \\
-a \nu = \kappa^\nu \Omega^M \nu & p^1 \text{ coeff.} \quad \text{--- (4)} \\
m^2 - k \Omega k = m^2 - a^2 & p^0 \text{ coeff.} \quad \text{--- (5)}
\end{cases}
\]

To find $\kappa^\nu$, need to solve equations (4) for it.

$\Rightarrow$ Need $\Omega^{-1}$

Claim: $\Omega^{-1} = \delta + \frac{ff}{1-f^2}$ \quad \text{--- (6)}
Claim: \( \Omega^{-1} = S + \frac{ff'}{1-f'^2} \) \( - (6) \)

To check, set \( \Omega^{-1} = S + Bff' \) for unknown \( B \)
and require that \( \Omega \Omega^{-1} = 1 \)

Using equation (4), we have

\[
\begin{align*}
(K \Omega^{-1}) \Omega^{-1} & = \kappa = -a \Omega^{-1} \\
& = -a \left( S + \frac{ff'}{1-f'^2} \right) \\
& = -a - \frac{(a \cdot f) f}{1-f'^2} \\
\kappa & = -a - \frac{(a \cdot f) f}{1-f'^2} \quad - (7)
\end{align*}
\]

Use (5) and (7) to get \( \mu \):

\[
(5) \Rightarrow \mu^2 = m^2 - a^2 + \frac{K \Omega \kappa}{\kappa} \\
& = m^2 - a^2 - a \cdot \kappa \\
& = m^2 - a^2 + a \left( a + \frac{(a \cdot f) f}{1-f'^2} \right) \\
\mu^2 & = m^2 + \frac{(a \cdot f)^2}{1-f'^2} \quad - (8)
\]

Now substitute the results found for \( \Omega, \kappa, \) and \( \mu \) into the Lagrange function for the quadratic case:

\[
\mathcal{L}(u) = \pm \mu \sqrt{u \Omega^{-1} u} + \kappa \cdot u
\]
is indeed homogeneous of degree one in the four velocity $u^\nu$.

Limits of $i$ and $ii$:

Notes

i. $L(u)$ is indeed homogeneous of degree one in the four velocity $u^\nu$

ii. Limits of $f \to 0$ and $a \to 0$:

\[
L(u; a) = \mp m \sqrt{u^2} - a \cdot u
\]

\[
L(u; f) = \mp m \sqrt{u^2 + \frac{(f \cdot u)^2}{1 - f^2}}
\]

iii. The mass term is altered only when both $a$ and $f$ are nonzero, through the presence of $f \cdot a$ coupling.

iv. The most general quadratic case for the minimal SME is "face" case.

v. Exercise: Find the Lagrange function for $f, a, e$. Solution is equation (8) and (9) of PLB 693, 443 (2010) [arXiv:1008.5062]

vi. For $c_{\mu\nu}$ coefficient, $\Omega^{-1}$ is an infinite series. Useful to decompose into symmetric and antisymmetric parts: $c_S$ and $c_A$. See discussion in above ref.
4. Quartic dispersion relations (restricted case)

For minimal Lorentz violation corresponding to quartic dispersion relations, concise/clean Lagrange functions are known for some special cases, which include

i. the $b_\nu$ coefficient
ii. the $H_{\mu\nu}$ coefficient with $Y = 0$
iii. the antisymmetric part of the $d_{\mu\nu}$ coefficient, with $Y = 0$

The Lagrange function for these cases and a few others can be found using the following result:

Claim: if the dispersion relation has the form

$$\left[(\rho + \kappa) \Omega \left( p + \kappa \right) - \mu \right]^2 = 4 (\rho + \kappa) S (p + \kappa)$$

and:

$$(S \Omega^{-1})^2 = \tilde{S} (S \Omega^{-1})$$

then:

$$L = - (\mu^2 + \tilde{S})^{1/2} \sqrt{u \Omega^{-1} u} - \sqrt{u \Omega^{-1} S \Omega^{-1} u} + \kappa \cdot u$$

Notes

i. The quantities $\Omega, \kappa,$ and $\mu$ are as for quadratic case.
ii. The quantity $S_{\mu\nu}$ is constant, and $S \to 0$ in Lorentz-preserving case.
iii. The extra condition simplifies $S$; only one nonzero eigenvalue is permitted
iv. See ref. below to explain the construction

Example: $b$ coefficient

Dispersion relation (found earlier):

\[
(p^2 - m^2 - b^2)^2 = 4(b \cdot p)^2 - 4b^2 p^2 \quad \quad (1)
\]

Can match to the template ...

\[
[(p + \kappa) \Omega(p + \kappa) - \mu]^2 = 4(p + \kappa) S(p + \kappa) \quad \quad (2)
\]

Calculations: RHS of (1):

\[
4(b \cdot p)^2 - 4b^2 p^2 = 4[p \cdot b)(b \cdot p) - p b^2 \rho] = 4 p (b b - b^2 S) p = b^2 b \rho
\]

So \[\kappa = 0\]

\[\Omega M = S M\]

\[\mu^2 = m^2 + b^2\]

\[S M = b^2 b - b^2 S\]

Calculations: Must verify that $(S \Omega^{-1})^2 = S(S \Omega^{-1})$

for some number $\delta$:

Note that $\Omega = \delta$, so $\Omega^{-1} = \delta$ (identity matrix)

So \[S \Omega^{-1} = S = bb - b^2 S\]

Consider \[S^2 = (bb - b^2 S)(bb - b^2 S)\]
Using the template, we can write down a Lagrange function for the given dispersion relation:

\[ L = -\left(\mu^2 + \bar{S}\right)^{1/2} \sqrt{u \Omega^{-1} u} + \sqrt{u \Omega^{-1} \Sigma \Omega^{-1} u} + k \cdot u \]

\[ \mu^2 = m^2 + b^2 \]

\[ \bar{S} = -b^2 \]

Note:

i. As expected, \( L \) is homogeneous of degree 1 in the 4-velocity

ii. In the quantum case, \( b_v \) couples to spin. Here, the two solutions are, in some sense, spin up and down.

Exercises

i. Find \( L(u) \) for \( H_{\mu\nu} \) with \( Y = 0 \). Solution is in section III.B of reference below.

ii. Find \( L(u) \) for antisymmetric \( d_{\mu\nu} \) with \( Y = 0 \). Solution is in section III.A of reference below.

iii. Confirm that \( L(u; b, f) \) cannot be found from the quartic template because the 'extra condition' fails.

7. References

References used specifically for this talk:

**Flat spacetime:**

**Curved spacetime:**
- Kostelecký, Russell, Tso, PLB 716, 470 (2012), [arXiv:1209.0750] "Bipartite Riemann-Finsler geometry and Lorentz violation"