Appendix: Why is e here?

This appendix gives a partial answer to the question: Why does light decrease as a negative exponential? A complete answer from first principles of photons scattering from spherical particles is very difficult. But the following provides an answer from the observation that a constant proportion of light is attenuated in a series of finite layers to the idea that there are an infinite number of layers (continuous medium).

Figure 1 shows the necessary geometry.

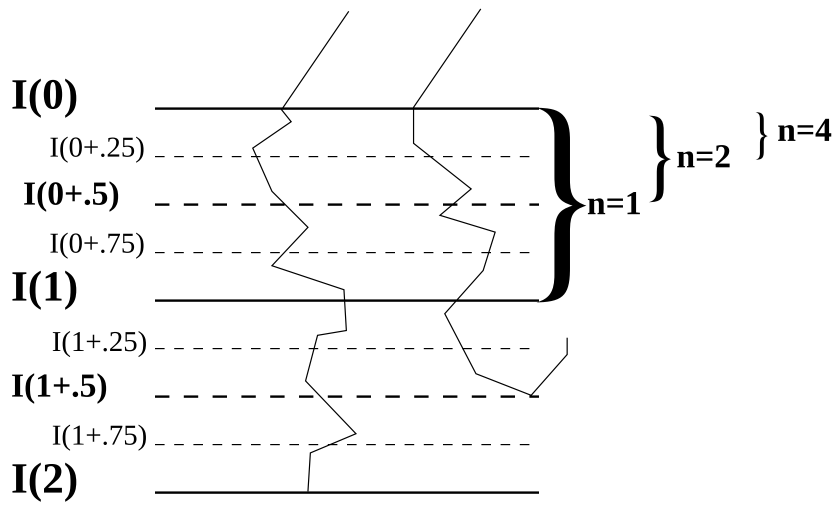

FIGURE 1: Imaginary layers in a continuous medium with light attenuation. Horizontal lines are imaginary layers; the two irregular lines are paths of 2 photons.

Assume light enters a medium at the top and is attenuated as photons move through the medium and are absorbed or scattered by the particles of the medium. Assume that each major layer $(I(0), I(1), I(2))$ attenuates the light entering by the same constant fraction R:

$$I_{i+1} = I_i R = I_i(1-q) \tag{1}$$

where q is the fraction attenuated.

This works fine if the medium really is discrete and light passes through the space in jumps. But water (and real tree canopies) is continuous and light is continuously removed from the medium. What equation describes this situation?

To answer this, we subdivide the medium into progressively more layers, and we reason from finitely many to an infinite number of layers. The number e is the result of doing this as follows.

Eqn 1 describes light attenuation over one large finite layer. Divide the layer in half so that now the interval is composed of $n = 2$ sub-layers (Fig. 1). In this geometry, it takes 2 steps to compute the light at layer 1:

$$\begin{aligned} I(0+.5) &= I(0)(1-q/2) \\ I(1) &= I(0.5)(1-q/2) \end{aligned}$$

Since $I(0.5)$ in the second equation is defined in the one above, we substitute the first equation for $I(0.5)$:

$$I(1) = [I(0)(1 - q/2)](1 - q/2) = I(0)(1 - q/2)^2$$

If we continue to subdivide each of the 2 sub-layers into 2 subsublayers ($n = 4$, Fig. 1), we have for 1 subsublayer

$$I(0.25) = I(0)(1 - q/4)$$

and for all 4 subsublayers

$$I(1) = I(0)(1 - q/4)^4$$

Extending this process to n layers

$$I(i + 1) = I(i)(1 - q/n)^n \quad (2)$$

What happens when $n \to \infty$? First, we need one of those little tricks that only a mathematician could dream up. We will multiply the power in Eqn 2 by 1, in this special form: $(-a/-a)$:

$$I(i + 1) = I(i)(1 - q/n)^{n(-a/-a)}$$

After re-arranging:

$$I(i + 1) = I(i)\left[(1 - q/n)^{-n/a}\right]^{-a}$$

Now as $n \to \infty$, $q/n \to 1/n$ and $-n/a \to -n$, so in the limit $n \to \infty$

$$I(i + 1) = I(i)\left[(1 - 1/n)^{-n}\right]^{-a} \quad (3)$$

The question now is: What is the limit of $(1 - 1/n)^{-n}$? Let

$$V_n = \left(1 - \frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^{n}$$

let $n - 1 = m$, so that

$$V_m = \left(\frac{m+1}{m}\right)^{m+1} = \left(1 + \frac{1}{m}\right)^{m+1} = \left[\left(1 + \frac{1}{m}\right)^{m}\right]\left(1 + \frac{1}{m}\right)$$

As $n \to \infty$, so does m. In the last equality on the right, the limit of the rightmost expression is 1 ($1/m \to 0$). The other expression in square brackets is more interesting. If you substitute progressively larger integers for m, you get this series, letting y be the expression in brackets:

m	1	10	500	1000	10,000
y	2	2.59374	2.71557	2.71692	2.71815

The expression y evidently has a definite limit, and we call this number e. So, the series V_m and V_n both converge on e. As a result, in the limit as the number of sub-layers goes to ∞ and substituting the limit (e) in Eqn 3 we have:

$$I(i + 1) = I(i)e^{-a}$$

And since this equation can be used recursively:

$$I(i + 2) = I(i + 1)e^{-a} = I(i)e^{-a2}$$

or, for any interval from 0 to i:

$$I(i) = I(0)e^{-ai}$$

where i is the total distance over which attenuation occurs. This is the Beer-Lambert Law.