Adapting Perception, Action, and Technology for Mathematical Reasoning

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RUNNING HEAD: Adapting for Mathematics

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Abstract

Formal mathematical reasoning provides an illuminating test case for understanding how humans can think about things that they did not evolve to comprehend. People engage in algebraic reasoning by 1) creating new assemblies of perception and action routines that evolved originally for other purposes (reuse), 2) adapting those routines to better fit the formal requirements of mathematics (adaptation), and 3) designing cultural tools that mesh well with our perception-action routines to create cognitive systems capable of mathematical reasoning (invention). We describe evidence that a major component of proficiency at algebraic reasoning is Rigged Up Perception-Action Systems (RUPAS), via which originally demanding, strategically-controlled cognitive tasks are converted into learned, automatically executed perception and action routines. Informed by RUPAS, we have designed, implemented, and partially assessed a computer-based algebra tutoring system called Graspable Math with an aim toward training learners to develop perception-action routines that are intuitive, efficient, and mathematically valid.

Key words: mathematics, symbolic reasoning, embodied cognition, perception, action

Introduction

Culture changes more rapidly than human brains evolve. As a result, we must perform culturally important cognitive feats using brains that evolved for other purposes. Written language and mathematical reasoning have probably existed for less than 6000 years, with examples dating back to the Sumerians in Mesopotamia. And yet, specific neural regions are implicated in reading and mathematics, despite the relative youth of these activities. The visual word form area in the left fusiform gyrus, for instance, is implicated in reading (Dehaene & Cohen, 2011). Visual number forms are processed in the inferior temporal gyrus and anterior to the temporo-occipital incisure (Shum et al., 2013), and the intraparietal sulcus and prefrontal cortex have been identified for their contributions to mathematical cognition (Amalric & Dehaene, 2016). How do these brain regions come to handle these culturally specific cognitive functions if literacy and mathematics have not been around long enough to have had a major impact on the evolution of brain structure?

Our answer to this question involves three components. First, people solve new cognitive tasks by reusing brain regions that evolved for other purposes (Anderson, 2015). Through reuse, new circuits are cobbled out of preexisting parts, allowing humans to achieve surprising intellective flexibility, much like Do-It-Yourself artists reuse trash to create dresses or inventors combine car parts to create robots. A striking example of reuse is the recruitment by blind people of the sub-area of the visual cortex that, in sighted adults, is involved in constructing spatial representations from visual input—but instead using it to construct spatial representations from *auditory and tactile* input (Reiner et al., 2010). We do not wish to deny that brains may have evolved for relatively general purpose perceptual tasks such as novel object recognition, eye-hand coordination across changing tasks and bodies, and fine discriminations between newly important visual categories. Instead, our point is just that special- or general-purpose perceptual processes are assembled together in new arrangements to solve recent, culturally significant tasks.

Second, through life-long adaptation, the brain is sufficiently malleable that its parts become tuned, over development, to their new requirements. For example, brains of literates and illiterates are organized differently, with greater specialization for words in the literate than illiterate left fusiform gyrus, and greater specialization for faces in the illiterate than literate right fusiform gyrus (Dehaene et al., 2010). Third, since human culture evolves faster than brains, one expedient strategy to improve mental processing is through the invention of tools that better leverage our brains' evolved capacities. Written symbols for words, for instance, may have adapted over history to maximize the efficiency by which they can be processed by the human visual systems (Changizi & Shimojo, 2005).

We will argue that all three of these kinds of flexibility are at work in mathematical reasoning generally and algebra in particular. We propose a hypothesis we call RUPAS (Rigged Up Perception-Action Systems), which states that an important way to efficiently perform sophisticated cognitive tasks is to convert originally demanding, strategically-controlled operations into learned, automatically deployed perception and action routines. "Rigging up" is intended in the sense of "flexibly assembling needed equipment from materials at hand." Algebra is one of the clearest cases of widespread symbolic reasoning in all human cognition (Anderson, 2007). If even algebraic reasoning relies on neural circuits evolved for perception and action—adapted individually and cobbled together—then almost any cognitive task is a likely candidate for perception-action grounding.

Employing Visual Grouping Routines for Algebra

Some of the aspects of algebraic reasoning that seem to be the most abstract turn out to be deeply perceptual. The same human visual system that identifies cars, guides reaches to coffee mugs, and organizes a wolf habitat at the zoo into a hierarchy of animals, families, and packs is put to work when processing algebraic notation — identifying symbols, guiding algebraic transformations, and organizing mathematical expressions into groups. Visual grouping processes allow some math experts to effortlessly perceive valid mathematical groupings: correctly parsing 3×b+c×4 as (3×b)+(c×4) instead of as 3×(b+c)×4; calculating (4+2)×(3+1)/(4-2)×(3-1) as 24 rather than 6. If you thought the answer was 6 — with an intermediate result of (6×4)/(2×2) — then evaluate the expression in a computer math program like R, Python, or even Google search, and then take a moment to reflect on what your incorrect answer suggests about the impact of visual symmetry and spatial balance on the perception of mathematical groups.

To study this influence of perceptual grouping on mathematics, we had undergraduates judge whether an algebraic equality was necessarily true (Landy & Goldstone, 2007a). We were interested in whether visual groupings could override participants' knowledge of the order-of-precedence rules in algebra (e.g., multiplication precedes addition). We tested this by using perceptual features like spatial proximity to encourage perceptual groupings that were either congruent or incongruent with the formal order of precedence, as shown in Figure 1. For example, when solving an expression like '2 + 3   ×   5,' the physical spacing is closer between the 2 and the 3 than between the 3 and 5. This spacing is incongruent with the order-of-precedence rule that dictates that 3 and 5 should be multiplied first, before the product is added to 2. This incongruent spacing might lead people to first add the 2 and 3, yielding an incorrect answer of 25 instead of the correct answer of 17. As predicted, validity judgments were swayed by formally irrelevant perceptual features. For instance, when order of operations was incongruent rather than congruent with the perceptual grouping suggested by spatial proximity, participants made four times as many errors.

Participants continued to show this influence of task-irrelevant perceptual cues despite receiving trial-by-trial correctness feedback. This suggests that sensitivity to perceptual groups is automatic or at least resistant to strategic, feedback-dependent control processes. Other research (Landy & Goldstone, 2010) indicates that people are heavily influenced by groupings based on perceptual properties when performing not only algebra but simple arithmetic as well. The influence of spacing persists even when the mathematical notation is only presented briefly and then masked, and when verbal encoding is encouraged (Rivera & Garrigan, 2016), and these perceptual effects combine with traditional order-of-operations rules in compound expressions (Jiang, Cooper, &, Alibali, 2014). Effective training procedures for teaching math and science often deemphasize explicit, verbalizeable knowledge, instead focusing on perceptual skill learning through adaptive training (Kellman, Massey, & Son, 2010; Mettler & Kellman, 2014).

Adapting Perception and Action Routines

The preceding examples illustrate ways in which algebraic reasoning with symbolic notations relies on perception. A lingering worry is that perceptual strategies may lead us astray. Philosophers and psychologists have argued that mathematical and scientific sophistication involves minimizing reliance on perception (Quine, 1977). This worry is allayed when we appreciate the adaptability of our perceptual systems (Goldstone & Barsalou, 1998; Goldstone, Landy, & Son, 2010). We rig up our perceptual systems so that they do the Right Thing, formally speaking (Goldstone, de Leeuw, & Landy, 2015; Kellman, Massey, & Son, 2010). Perceptual learning is a general mechanism by which perception is adapted and attention is directed to relevant environmental structures (Gibson, 1969). Moreover, perceptual learning has been explicitly implicated in improving the selectivity and fluency with which humans extract important regularities in mathematics (Kellman & Massey, 2013).

As an example of perceptual learning to support mathematics, the visual-attentional system will give higher priority to notational operators that have higher precedence. For instance, the notational symbol for multiplication ("×") attracts more attention than the lower precedence addition operator ("+"), even when participants do not have to solve mathematical problems. People who know algebra show earlier and longer eye fixations to "×"s than "+"s in the context of math problems (Landy, Jones, & Goldstone, 2008). When asked simply to determine the center operator in expressions like "4 × 3 + 5 × 2," participants' attention is diverted to the peripheral "×"s: performance is worse than both "4 + 3 + 5 + 2" and "4 + 3 × 5 + 2" trials (Goldstone, Landy, & Son, 2010). Thus, in the competition for attention, the operator for multiplication wins over the operator for addition. This is not due simply to specific perceptual properties of the symbols themselves. Similar asymmetries are found when participants are trained with novel operators with arbitrary orders of precedence. These results suggest that attention is attuned to formal mathematical knowledge. We attend where we should if we are to act in accordance with the order of precedence.

Given its adaptability, perception supports high-level cognition far more effectively than proponents of low-level, sensory accounts of perception might surmise. Perception is frequently relational, complex, and structural (Kellman & Massey, 2013). Adherents of a "Perception is (often) misleading" view might predict that reliance on superficial perceptual cues should diminish with mathematical maturity. Surprisingly, in several cases we have found the opposite pattern. For instance, older children rely more on physical spacing as a cue to perceptual organization than younger children (Braithwaite, Goldstone, van der Maas, & Landy, 2016). We analyzed the solutions of 65,856 Dutch children, from age 8 to 12 years, for simple math problems in which physical spacing was manipulated to be either congruent (2 + 7×5) or incongruent (2+7 × 5) with the formally defined order of operations, where multiplications are executed before additions.

Incongruent spacing increased the selection of incorrect foils, such as responding that $2+7 \times 5$ equals 45, as if the problem was parsed incorrectly as $(2+7) \times 5$. If learning to reason mathematically involves learning to overrule potentially misleading perception, then this effect should decrease over time—but the difference in accuracy between incongruently and congruently spaced problems actually increased with age and math experience. Becoming proficient in math may involve learning to rely more, not less, on perceptual grouping routines.

Learning mathematics changes not only the visual salience of notational objects, but also how they are grouped together. When different parts of the visual world are perceived by an observer as belonging to the same object, comparison of the parts is more effective than when they seem to come from different objects (Zemel, Behrmann, Mozer, & Bavelier, 2002). We applied this same logic to algebraic objects such as those shown in Figure 2. When tasked with deciding whether two symbols were the same or different color, our observers were more accurate when the symbols belonged to a high precedence algebraic object, such as the "$m \times j$" term of "$v + m \times j + a$" rather than "$j + a$" (Marghetis, Landy, & Goldstone, 2016). However, this benefit of within-group over between-group visual comparisons was only found for observers who demonstrated understanding of the order-of-precedence rules in algebra. Moreover, increased object-based attention for algebraic objects was associated with a better ability to evaluate algebraic validity. This suggests, again, that mathematical proficiency may rely on adapting perception to satisfy formal, mathematical requirements.

Mathematical sophistication is associated not just with increased reliance on spatially constrained perceptual routines, but also spatially constrained action routines. Rather than converting algebraic notation into a non-spatial, completely formal internal representation, proficient reasoners often preserve the spatial format of algebraic notation and apply spatial transformations within this notation space. Take a moment to try solving for b in the problem $2 \times b = 14$. One spatial transformation you may have used for this problem is dynamic transposition, in which you imagined moving the 2 from the left side of the equality to the right side, whereupon you moved it to the denominator of a 14/2 quotient. Although such spatial strategies are highly intuitive, it is noteworthy that they do not appear in most prominent models of algebra (e.g., Anderson, 2007).

To measure if and when participants adopt a spatial transposition strategy for solving simple algebraic equations, equations were superimposed on top of vertical bars that moved continuously to either the left or right (Figure 3; Goldstone, Landy, & Son, 2010). To isolate y on the left side of the equation "4 * y + 8 = 24," for instance, the 4 and 8 must be moved to the right side; a rightward motion of the bars would be thus be compatible with a transposition strategy. For the formally-equivalent equation "24 = 4 * y + 8," however, the same rightward motion would be incompatible. Participants solved the equations more accurately when the grating motion was compatible with transposition, suggesting that participants were simulating transpositions within the spatial world of the notation. This effect was stronger among participants who had taken advanced mathematics courses such as calculus, compared to students with less math experience. This imagined motion strategy, therefore, is not a crutch that students adopt while learning but abandon as their sophistication increases, but a "smart" strategy that students come to adopt through experience with formal notations. Learned perceptual routines are not at odds with strong mathematical reasoning. They are often the means by which strong mathematical reasoning becomes possible. It is a smart strategy to take advantage of the scaffolding provided by space, using it as a canvas on which to project transforming motions.

Adapting Cultural Products to Fit Perception and Action

The previous sections have described ways in which mathematical experience tunes our perceptual systems. We also tune our notational systems to fit our perceptual systems. Much of the history of mathematical notation is one of changing notations over time to better fit human perceptual systems

(Cajori, 1928). For example, the historic shift from representing "3 times the variable b plus 5" as "3×b+5," to later representing it as "3•b+5," and more recently as "3b+5," represents a consistent shift toward decreasing the spacing between operands that should be combined together earlier. This adaptive redesigning of mathematical notations also occurs on shorter timescales. Landy and Goldstone (2007b) asked participants to write symbolic mathematical expressions for equations expressed in English such as "nine plus twelve equals nine plus three times four" (see Figure 4). From these expressions, we measured the physical space around the different operators. On average, the physical spacing was largest around "=", consistent with its role as the highest level structural grouping for the equation. The physical spacing was larger around the "+" than around the "×" in equations that had both of these operators. Our account of this result is that people produce notations that their own perceptual systems are well prepared to process. Students discover this adaptive tactic even though most textbooks do not use physical spacing to help them form useful perceptual groups in algebra. In this way, we create notations that are processed aptly by our rigged-up perceptual systems—one more reason why perceptual systems should often be trusted rather than trumped. Well-designed tools can turn error-prone conceptual tasks into robust perceptual ones (Hutchins, 1995).

We believe this slow cultural evolution of notation forms can be accelerated dramatically by modern computational technology. Interactive technologies informed by cognitive science can be harnessed to improve the fit between our cultural tools for mathematics and human perception-action routines. Our investigations of RUPAS have led us to implement an interactive algebra notation system called Graspable Math (GM: http://graspablemath.com) that allows users to interact in real-time with math notation using intuitive and trained perception-action processes. The primary rationale for this system is that people often come to be proficient reasoners in mathematics not by ignoring perception, but by educating it (Goldstone, de Leeuw, & Landy; 2015; Goldstone, Landy, & Son, 2010; Landy, Allen, & Zednik, 2014). Perception and action routines are subject to intrinsic constraints such as spatial and temporal contiguity (e.g. one cannot easily attend to two objects without attending the object in between them), which the notation must honor. However, these constraints can still be honored by a variety of perception-action routines that can be flexibly shaped by real-time, dynamic feedback. Consequently, our interactive notation support system is designed to generate experiences that develop effective and intuitive perception and action routines (Ottmar, Landy, Weitnauer, & Goldstone, 2015).

GM provides a concrete model of how the content of algebraic transformations can be supported by perception and action routines. Rather than translating algebraic objects into other concrete objects such as blocks, coins, rods, or balance beams, GM is based on the appreciation that algebraic objects are also concrete, albeit notational objects. Algebraic notation has been crafted over time to fit human perception and action systems, and computer technology can allow notation to fit humans even better, by devising interactive notation that dynamically responds to users' gestural movements.

Figure 5 shows some common actions related to mathematical reasoning that are supported by GM. Each of them is a physical and spatial action that results in valid mathematical transformations. For example, if constrained properly, the operation of spatially swapping factors (e.g., $A \times B = B \times A$) corresponds to the formal, commutative property of multiplication. Likewise, spatial transposition can also be specified in a way that makes it mathematically valid. Still, some teachers resist teaching transposition, viewing it to be an illegitimate algebraic transformation. They object, "You shouldn't teach students that they can just move the 2 of $y-2=5$ to the right side and change its sign. Students should go through the axiomatically justified steps of adding 2 to both sides of the equation, yielding $y-2+2=5+2$, and then simplifying to $y=5+2$." To this objection, we respond that the teacher's preferred solution is <u>one</u> justifiable transformation pathway, but mathematics is rich enough to permit multiple pathways to produce valid mathematical reasoning. The spatial transformations shown in Figure 5 provide an alternative approach to the traditional axiomatization provided by Euclid. For example, Euclid's second axiom states: if two quantities are equal and an equal amount is added to each, they are still equal. Our alternative, spatial transformation is more psychologically intuitive because it has been designed to be processed efficiently by human perception-action systems. It is also more efficient, given that students can isolate the y in "y-2=5" with two fewer transformations. Finally, it is conceptually evocative. As the -2 crosses the equal sign and becomes +2, learners experience viscerally a deep mathematical relation: if Y is equal to a one-to-one function of X, then X is equal to the inverse of that function applied to Y. With respect to the first point about GM being "conceptually evocative," in our rewrite we try to be open about the value of different mathematical reasoning systems. We write: Euclid's second axiom and spatial transposition are coupled to different, deep mathematical insights. Doing the "same thing" to two expressions that are equal preserves their equality is coupled with Euclid's second axiom, and $x=f(a) \Leftrightarrow a=f^{-1}(x)$ is coupled with transposition. We suspect that both insights are valuable and so we advocate a system, like Graspable Math, that is ecumenical enough to support both transformation pathways.

The coupling of different concrete actions with different conceptual construals raises the large issue of what is relation between procedural and conceptual understandings in math. Education research often divides knowledge into conceptual knowledge of abstract and general principles versus procedural knowledge of the steps or actions needed to accomplish a goal (Baroody, Feil, & Johnson, 2007). One of the motivations for this division is teachers' common frustration that their student seem to just want to know the steps needed to calculate the solution to a problem without taking the time to develop flexible generative models or genuine understandings of why these steps lead to the right answer. This phenomenon certainly occurs, but our experiences with developing systems for instruction in mathematical reasoning have led us to believe that procedural and conceptual knowledge mutually support and inform one another (see also Rittle-Johnson, Schneider,

& Star, 2015). We resist characterizing procedural knowledge as necessarily inflexible or "rote." The promise of technologies like GM is that students will develop apt intuitions about what are the right ways to perceive or act upon mathematical expressions. These introspections are steered by analytic reasoning, but once established, can act as a catalyst for subsequent analytic insights.

Conclusions

It is widely assumed that, as it develops, mathematical reasoning shifts toward abstraction. But our initial observations of mathematicians "in the wild" suggest that their reasoning depends on spatial perceptual grouping strategies and actions over space. Sophisticated reasoners are at least as likely to employ concrete actions as novices — they just apply them more efficiently and felicitously. This realization has directly informed our development of new technological tools for doing and teaching mathematics.

Algebraic reasoning provides an excellent example of how our brains come to be able to do things for which that they were never prepared by evolution. To harness the powers of math to make valid real world inferences, we deploy three strategies: (1) <u>reusing</u> existing mental processes that developed over millions of years for non-algebraic purposes by cobbling them together as novel routines for newfangled, algebraic purposes; (2) <u>adapting</u> those perception-action routines to fit the formal requirements of mathematics; and (3) <u>inventing</u> cultural tools that better fit the cognitive constraints and potential of perception and action. This account intertwines three temporal scales of adaptation from long to short: the slow evolution of bodies, including brains, to accommodate persisting biological needs; the creation of cultural innovations like pencils, calculators, and Graspable Math that can be joined with people to create cognitive systems with capabilities out of the reach of independent humans; and rigging up our perception and action systems (RUPAS) over our lifespan. These last two mechanisms offer particular promise for mutually reinforcing adaptations, because they occur on similar timescales—and those timescales are becoming more similar as our tools for creating tools improve, further accelerating the rate of cultural change. Humans adapt to technology while, ever more rapidly, technology can adapt to us.

Notes

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Figure Captions

Figure 1. Samples from three experiments reported by Landy and Goldstone (2007a). Participants were asked to verify whether an equation is necessarily true. Grouping suggested by factors such as spatial proximity (top), connectedness of surrounding graphical forms (middle), and proximity in the alphabet (bottom), could be either congruent or incongruent with the order of precedence of arithmetical operators (e.g. multiplications are calculated before additions). When perceptual groups are congruent with formal order of precedence then validity judgments are much more accurate than when they are incongruent. For instance, in the top equality, f + z * t + b is not necessarily equal to t + b * f + z, but participants often decided otherwise in the incongruent version. This is presumably because the narrow spacing around the "+" signs encouraged them to form "f + z" and "t + b" units, which exist on both sides of the equation.

Figure 2. Sample stimuli (Panel 2A) and results (Panel 2B) from Marghetis, Landy, and Goldstone (2016). When participants were asked to judge whether two colored symbols in a briefly presented mathematical expression shared the same color or possessed different colors, they were more accurate (higher d') when the symbols belonged to a proper mathematical object rather than straddled two objects. This effect, however, was only found for "Syntax Knowers" — participants who demonstrated understanding of the order of precedence in algebra whereby multiplications are calculated before additions.

Figure 3. As participants solved for the variable in equations like the above, a vertically oriented grating continuously moved either to the left or to the right. Although irrelevant for the task, when the movement of the grating was compatible with the postulated imagined movements of the numbers required by spatial transposition, participants were more accurate.

Figure 4. Written equations use space to indicate the formal precedence of arithmetic operations. Here, a hand-drawn equation places less space around the '×' symbol — which represents a high-precedence operation — and places relatively less space around the '=' and '+' symbols. (This particular equation was produced by a participant in Landy and Goldstone, 2007b.) Dashed squares illustrate relative spacing.

Figure 5. Examples of physical transformations within notational space. A variety of physical operations are often employed in a cognitively efficient and valid fashion by mathematical reasoners. When algebra is implemented with pen-and-paper, these operations are typically only simulated mentally, with the reasoner projecting their imagined operations onto the written equation, until eventually they rewrite the transformed equation. In a dynamic algebra system like Graspable Math, these operations can be performed literally by interacting with the equation.

Arrows indicate physical operations; faded symbols represent the initial state of the equation; green symbols represent the goal state; the red line indicates cancellation.

Figure 1

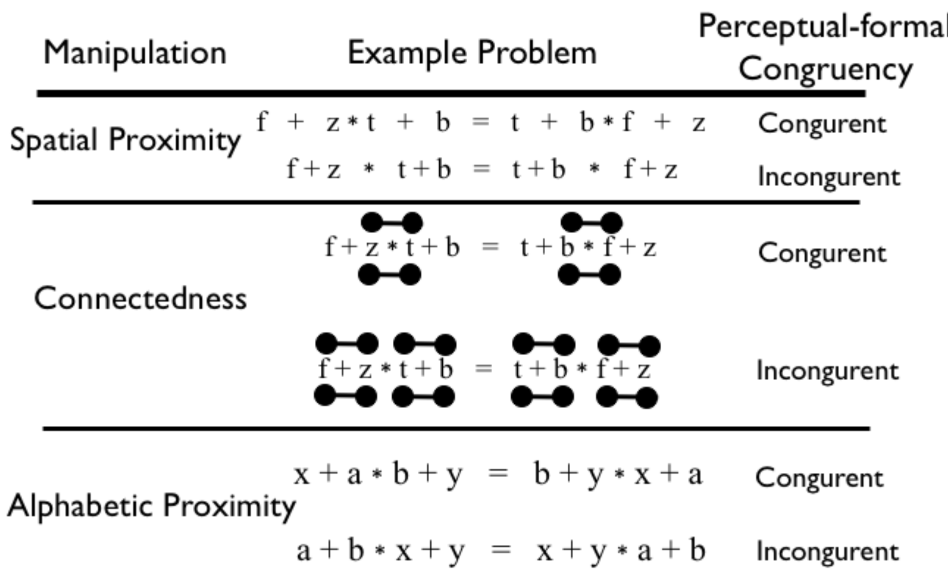
Manipulation
Example Problem
Perceptual-formal Congruency
Spatial Proximity
f + z * t + b = t + b * f + z
Congruent
f+z * t+b = t+b * f+z
Incongruent
Connectedness
f+z*t+b = t+b*f+z
Congruent
f+z*t+b = t+b*f+z
Incongruent
Alphabetic Proximity
x + a * b + y = b + y * x + a
Congruent
a + b * x + y = x + y * a + b
Incongruent

Figure 2

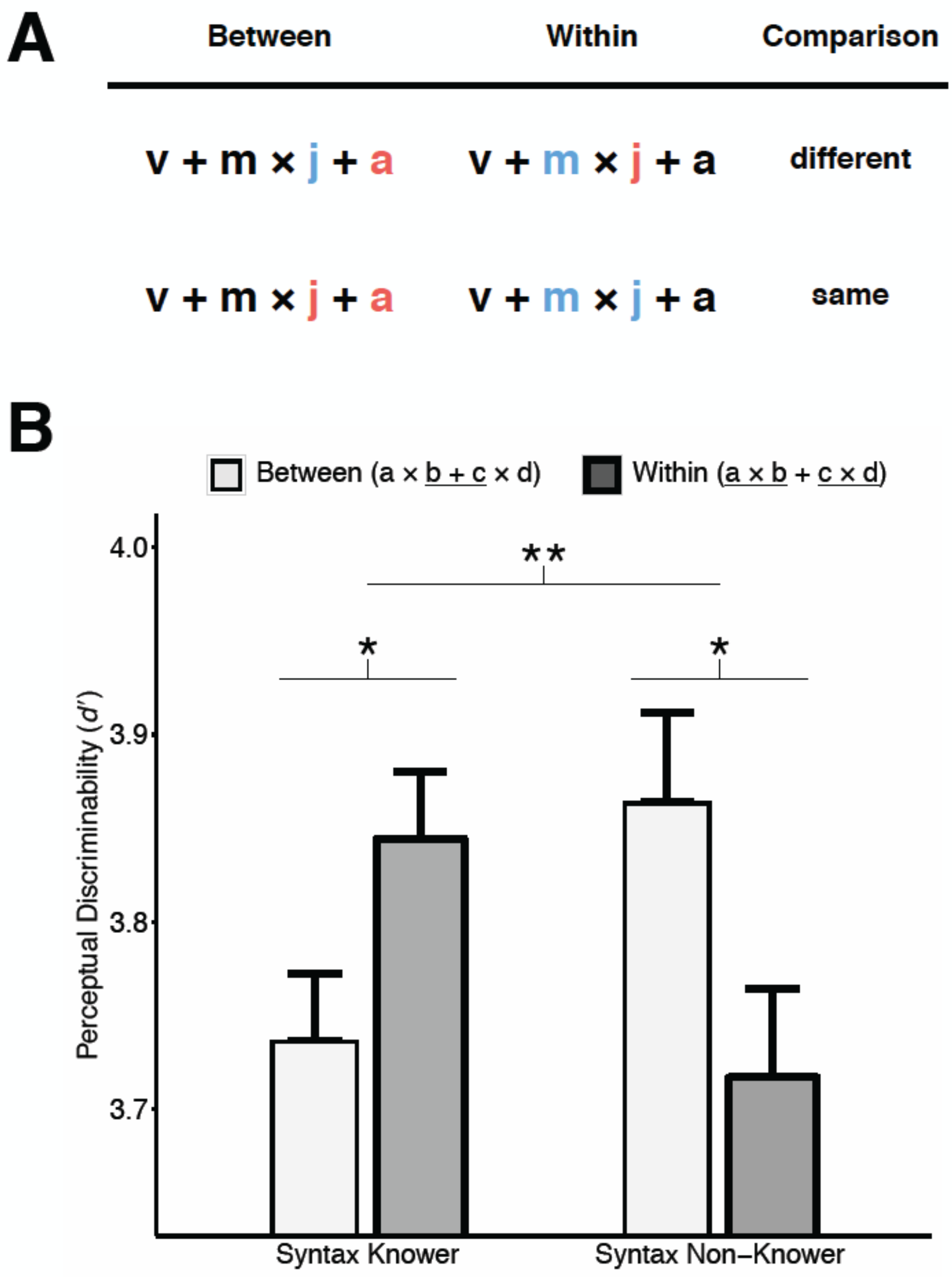
A
Between
Within
Comparison
v + m × j + a
v + m × j + a
different
v + m × j + a
v + m × j + a
same
B
Between (a × b + c × d)
Within (a × b + c × d)
Perceptual Discriminability (d')
4.0
3.9
3.8
3.7
**
*
*
Syntax Knower
Syntax Non–Knower

Figure 3

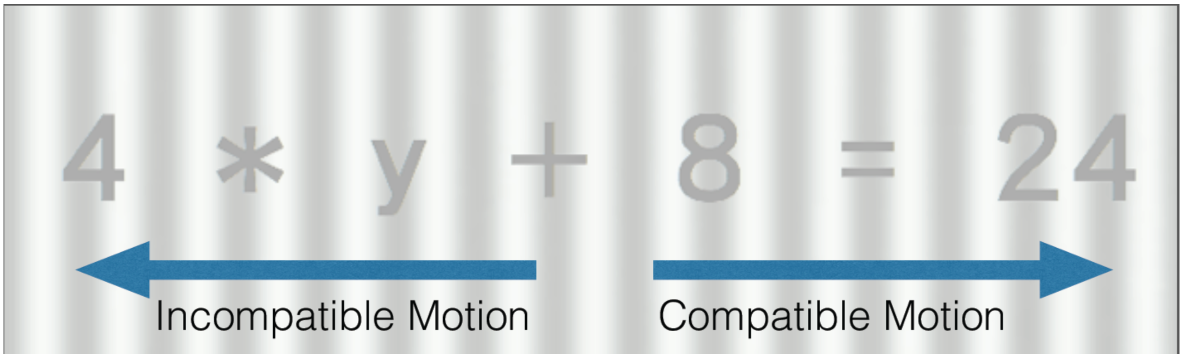

Figure 4

“Write the equation for: nine plus twelve equals nine plus three times four”

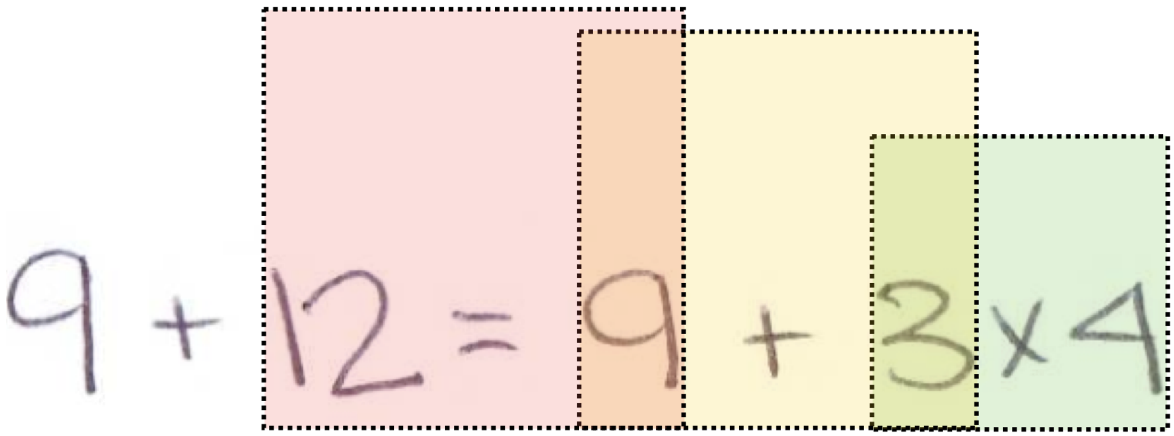

Figure 5

Spatial Transformation	Initial State	Transformed State
Swapping	$3 = (x - 6)\times(7 + y)$	$3 = (7 + y)\times(x - 6)$
Splitting	$y = a\times(5 + 7)$	$y = a\times(5a + 7a)$
Transposing	$x + 3 = 8$	$x + 3 = 8 - 3$
Simplifying	$y = 3 + 4 \times 7$	$y = 3 + \overset{28}{4 \times 7}$
Cancelling	$z = \frac{3x}{3y}$	$z = \frac{\cancel{3}x}{\cancel{3}y}$