# Modeling Mathematical Reasoning as Trained Perception-Action Procedures

Robert L. Goldstone, Indiana University

Erik Weitnauer, Indiana University

Erin R. Ottmar, Worcester Polytechnic Institute

Tyler Marghetis, Indiana University

David H. Landy, Indiana University

We have observed that when people engage in algebraic reasoning, they often perceptually and spatially transform algebraic notations directly rather than first converting the notation to an internal, non-spatial representation. We describe empirical evidence for spatial transformations, such as spatially compact grouping, transposition, spatially overlaid intermediate results, cancelling out, swapping, and splitting. This research has led us to understand domain models in mathematics as the deployment of trained and strategically crafted perceptual-motor processes working on grounded and strategically crafted notations. This approach to domain modeling has also motivated us to develop and assess an algebra tutoring system focused on helping students train their perception and action systems to coordinate with each other and formal mathematics. Overall, our laboratory and classroom investigations emphasize the interplay between explicit mathematical understandings and implicit perception-action training as having a high potential payoff for making learning more efficient, robust, and broadly applicable.

## INTRODUCTION

For the last several years our group has been involved in a project that can be construed in terms of exploring the relation of formal knowledge and perception. At first sight, formal cognition as seen in scientific and mathematical reasoning involves developing deep construals of phenomena that run counter to untutored perception. In fact, Quine (1977) considered a hallmark of advanced scientific thought to be that it no longer requires notions of overall perceptual similarity as the basis for its categories. The background rationale for this claim is that unanalyzed perceptual similarities may lead one astray. For example, marsupial wolves may closely resemble placental wolves, but they are evolutionarily rather distant cousins. In general, as a scientist or child (Carey, 2009) develops more complete, systematic knowledge about the reasons something has a property, then overall perceptual similarity becomes decreasingly relevant for generalizations.

While there is certainly justification for *opposing* principled understanding with superficial perception, we have been exploring the converse strategy of trying to *ground* scientific and mathematical reasoning in perceptual processing (see also Kellman, Massey, & Son, 2010). One reason for thinking that perception and formal thinking can be brought closer together is that we can train our perceptual processes to do the right thing, formally speaking. Perception and visual attention are highly educable processes. We can train our perception to process stimuli in an efficient manner for tasks that are important to us. An advantage of linking high-level processes to perception is that we can co-opt our neurologically large and phylogenically early perceptual areas – areas that are the result of millions of years of evolutionary research and development. Finally, there are suggestive correlations across individuals between perceptual and conceptual abilities (Goldstone & Barsalou, 1998). For example, schizophrenics have difficulty inhibiting both inappropriate thoughts and irrelevant attributes. Autistics often suffer from overly selective attentional processes, including hypersensitivity to sensory stimulation and overly narrow language generalization.

Based on these considerations, we have developed a hypothesis we call RUPAS (*Rigged Up Perception-Action Systems*) (Goldstone, de Leeuw, & Landy, 2015), which states that an important way to efficiently perform sophisticated cognitive tasks is to convert originally demanding, strategically-controlled operations into learned, automatically executed perception and action processes. These tasks can be understood as on par with the "visual routines" proposed by Shimon Ullman (1984) to account for how people extract information from a visual scene using processes such as shifting attentional focus, indexing items, tracing boundaries, and spreading activation from a point to the boundary of an area. In this chapter, we apply this general theoretical approach of exploring ways in which our sophisticated, formal reasoning abilities are grounded in perception and action to the specific knowledge domain of algebra. Algebra is one of the clearest case of symbolic reasoning in all human cognition (Anderson, 2007). Showing that perceptual factors influence even algebraic reasoning provides prima facie support for the premise that perception-action grounding cannot be ignored for almost any cognitive task.

## RELATED RESEARCH

To study the influence of perceptual grouping on mathematics, we gave undergraduate participants a task to judge whether an algebraic equality was necessarily true (Landy & Goldstone, 2007a). We were interested in whether perceptual and form-based groupings would be able to override participants' general knowledge of the order of precedence rules in algebra, according to which multiplications are executed before additions. We tested this by having perceptual grouping factors either consistent or inconsistent with order of precedence. For example, if shown the stimuli in the top row of Figure 1, participants would be asked to judge whether f + z * t + b is necessarily equal to t + b * f + z. In fact, this is not a valid equality. However, in the incongruent version of the physical spacing manipulation, the narrow spacing around the "+" signs might encourage participants to group the f and z together to form a "f + z" unit, as well as forming a "t + b" unit. If participants then match up these units on the left and right sides of the equation, they will find the same two units on the right side, leading them to respond "valid." As predicted, participants make 38% more errors on trials like this in which the formally determined order of operations is incongruent rather than congruent with the perceptual grouping suggested by the surround lines and circles. Other methods for manipulating perceptual groupings, like varying the connectedness of dots surrounding the mathematical expression (middle row) and proximity in alphabet (bottom row) also affect validity judgments. Participants continued to show large influences of grouping on equation verification even though they received trial-by-trial feedback. Feedback reduced, but did not eliminate the influence of these perceptual cues. This suggests that sensitivity to grouping is automatic or at least resistant to strategic, feedback-dependent control processes.

Other research (Landy & Goldstone, 2010) indicates that people are heavily influenced by groupings based on perceptual properties when performing not only algebra but simple arithmetic as well. Despite being reminded of, and verbally subscribing to, standard order of precedence rules, our college student participants are much more likely to calculate an incorrect solution value of 25 for "2+3  *  5  = ?" than "2  +  3*5 = ?," presumably because the narrow spacing around the "+" in the former case biases people to calculate 2+3 before they multiply by 5.

People not only respond to the perceptual cues contained within symbolic representations, but they also add perceptual cues when they construct their own symbolic representations. Landy and Goldstone (2010) asked participants to write symbolic mathematical expressions for equations expressed in English such as "nine plus twelve equals nine plus three times four." Figure 2 shows an example of one participant's symbolic expression. From these expressions, we measured the physical space around the different operators. On average, the physical spacing was largest around "=", consistent with its role as the highest level structural grouping for the equation. The physical spacing was larger around the "+" than around the "x" in equations that had both of these operators. Our account of this result is that people produce notations that their own perceptual systems are well prepared to process. One noteworthy aspect of this empirical result is that people are creating perceptual cues that help them do the formally right thing even when this activity is not modeled for them by textbooks. Our corpus analysis reveals that most mathematics textbooks depict equal spacing around multiplications and additions. So, even though books do not use physical spacing to help students form useful perceptual groups in this instance, students still engage in this practice. And this practice is actually a good indicator of mathematical skill. In fact, students who place wider physical spacing around lower, compared to higher, precedence operators also tend to produce the correct answer to math and logic problems (Landy & Goldstone, 2010). In this manner, we create notations that are processed aptly by our perceptual systems, and this is one of the reasons why perceptual systems should often be trusted rather than trumped.

## Rigged Up Perception Systems

The preceding examples illustrate ways in which we rely on perceptual processes to process symbolic notations. The lingering worry is that, as is the case with placental and marsupial wolves, or gold and pyrite (fools gold), appearances may be misleading. If mathematicians rely on superficial perceptual cues to decide how to process mathematical notation, won't they often be led astray?

One answer, described in the previous section, is that notations are not fixed and inflexible, but rather can be flexibly tuned to humans' perceptual systems because they are, after all, crafted by humans. This tuning occurs within an individual's lifetime, as with the case of physical spacing in students' written mathematical expressions, and also occurs on historic time scales. Much of the history of mathematical notation is one of changing over time to better fit human perceptual systems (Cajori, 1928). For example, the historic shift from representing "3 times the variable b plus 5" as "3×b+5," to later representing it as "3•b+5," and more recently as "3b+5," represents a consistent shift toward decreasing the spacing between operands that should be combined together earlier. Both individually and culturally speaking, we craft notations so that "superficial" perceptual cues are actually reliably indicators of correct reasoning, co-opting our perceptual systems to serve our cognitive needs.

As we tune our mathematical notations to fit our perceptual systems, we also tune our perceptual and attentional systems to fit math. People train their visual-attention processes to give higher priority to notational operators that have higher precedence. The operator for multiplication, "×," attracts attention more so than does the notational symbol for the lower precedence addition operator, "+." People who know algebra show earlier and longer eye fixations to "×"s than "+"s in the context of math problems (Landy, Jones, & Goldstone, 2008). Even when participants do not have to solve mathematical problems, their attention is automatically drawn toward the "×"s. When simply asked to determine what the center operator is for expressions like "4 × 3 + 5 × 2," participants' attention is diverted to the peripheral "×,"s as indicated by their inaccurate responses compared to "4 + 3 + 5 + 2" trials (Goldstone, Landy, & Son, 2010). The distracting influence of the peripheral operators is asymmetric as shown by the result that responding "×" to "4 + 3 × 5 + 2" is significantly easier than responding "+" in "4 × 3 + 5 × 2." That is, the operator for multiplication wins over the operator for addition in the competition for attention. This is not simply due to specific perceptual properties of "×" and "+" because similar asymmetries are found when participants are trained with novel operators with orders of precedence that are counterbalanced. The results suggest that a person's attention becomes automatically deployed to where it should be deployed to get them to act in accordance with the formal order of precedence in mathematics.

## Rigged Up Action Systems

Mathematical symbol systems would not be very valuable if the only thing we could do with symbols was to perceive them. In fact, we also transform symbols, deriving new implications and relations that lay dormant in their original form. What are the processes that are responsible for these transformations? One possibility is that symbolic transformations are executed internally using abstract representations. This account is tempting because notations like "$2 \times b = 14$" seem to be straightforwardly translatable into hierarchical mental representations like "$= ( \times (2, b), 14)$." From this representation, propositional transformation rules like "$= ( \times (a, b), c) \Rightarrow = ( b, \div (c,a))$" can be applied to solve for b. This kind of propositional transformation rule is powerful because of its generality and ability to operate on arbitrary inputs without any influence of their original spatial and perceptual properties (Newell & Simon, 1976).

However, as we have already seen, humans are indeed influenced by the spatial and perceptual properties of notations. Accordingly, the alternative account of symbolic transformation that we have pursued is to keep the symbolic form in its original spatial format, and apply simulated spatial transformations to this world of notations. For the $2 \times b = 14$ problem, one candidate transformation is spatial transposition, in which the 2 is moved from the left side of the equality to the right side, where upon it is moved to the denominator of a 14/2 quotient. This spatial movement might be executed literally, for instance using number tiles if they are at the reasoner's disposal. More often, they are executed in the reasoner's mind. Although this transposition operation is highly intuitive, expressed in language when we talk of solving the equation by "*moving* the 2 from one side to the other," it is noteworthy that this kind of spatial transformation does not appear in most leading models of algebra (e.g. Anderson, 2007).

Displays like the one shown in Figure 3 were devised to measure if and when participants adopt a spatial transposition strategy for solving simple algebraic equations. Equations were superimposed on top of a vertically oriented grating that continuously moved to either the left or right. The movement of the grating was either compatible or incompatible with the movement of numbers implicated by a transposition strategy. For the equation "4 * Y + 8 = 24" shown in Figure 3, a rightward motion of the grating would be compatible with transposition because, in order to isolate Y on the left side, the 4 and 8 must be moved to the right side. However, for the equation "24 = 4 * Y + 8," a rightward motion would be incompatible. Participants solved the equations more accurately when the grating motion was compatible with transposition.

The influence of background motion on algebraic solutions is consistent with a "visual routines" (Ullman, 1984) approach to mathematical cognition. According to this notion, people engage in dynamic, visual-spatial routines to perform perceptual computations. Of particular relevance to the perceptual learning aspect of this transposition routine, we also found that participants who have taken advanced mathematics courses such as calculus are more affected by the compatibility of the background motion than students with less math experience. Accordingly, we conclude that the imagined motion strategy is a smart strategy that students come to adopt through experience with formal notations, rather than a strategy that students initially use while learning, and then abandon as their sophistication increases. Learned perceptual routines are not at odds with strong mathematical reasoning. They are often the means by which strong mathematical reasoning becomes possible. It is a smart strategy to take advantage of the scaffolding provided by space, using it as a canvas on which to project transforming motions.

Another result consistent result with increasing use of space in notation with increasing mathematical sophistication is that older children rely more on physical spacing as a cue to perceptual organization than do younger children. Braithwaite, Goldstone, van der Maas, and Landy (2016) analyzed a corpus of 65,856 8-12 year old Dutch children's solutions to simple math problems in which physical spacing was manipulated to be either congruent or incongruent with the formally defined order of operations. For example, the physical spacing in "2+7 × 5" is incongruent with the rule that multiplications are executed before additions, whereas "2 + 7×5" is congruent. Incorrect answers like 70, the answer that would be produced if the problem was incorrectly organized as (2+7) ×5, were much more common with the incongruent spacing. The fact that the difference in accuracy between incongruently and congruently spaced problems increased with age and math experience is not expected under the notion (e.g. Vygotsky, 1962) that mathematical development involves a shift from informal mechanisms to formal rules and axioms. Instead, the study shows that reliance on informal mechanisms can sometimes systematically increase with age.

Figure 4 shows other common actions related to mathematical reasoning. Each of them is a physical and spatial action that nonetheless can be made to align perfectly with formally valid operations. For example, if properly constrained, the operation of spatially swapping factors is formally sanctioned by the commutative property of multiplication. Likewise, the intuitive act of cancelling out the two 3s in the bottom problem of Figure 4 can be formally sanctioned by a multiple step axiomatic derivation: $(3\times X)/(3\times Y) \Rightarrow (3/3)\times(X/Y) \Rightarrow 1\times(X/Y) \Rightarrow X/Y$. Future empirical work will be necessary to determine how often these actions are performed by mathematical reasoners and how effective they are. Our preliminary observations indicate that spatial actions like swapping A×B for B×A, splitting the a in a×(5+7) to form 5a + 7a, moving the 3 from the left side to the right side of the equality in X+3=8, simplifying 4×7 by projecting 28 on top of the original term, and canceling out the 3s in 3x/3y are commonplace and often times deployed effectively. While it is plausible and intuitive that mathematical reasoning should shift towards abstraction as it develops, our initial observations of mathematicians "in the wild" suggest that they are at least as likely to employ these kinds of spatially concrete transformations as are less sophisticated reasoners. Sophisticated reasoners still use concrete actions – they just apply them more efficiently and felicitously.

## DISCUSSION

On the basis of our laboratory investigations of rigged up perception and action routines for mathematical reasoning, we have implemented algebra tutoring software systems with a specific aim in mind: to help students rig up their perception and action systems for effectively processing algebraic notation and thinking mathematically. Currently, the most actively developed version of the system, named Graspable Math (http://graspablemath.com), allows students to interact in real-time with math notations using perception-action processes. The system is a natural outgrowth of our empirical findings suggesting that people come to be proficient reasoners in science and mathematics not by ignoring perception, but by educating it (Goldstone, Deleeuw, & Landy; 2015; Goldstone, Landy, & Son, 2010; Landy, Allen, & Zednik, 2014). Our intention is to construct a virtual sandbox for students to explore how algebra operates, and to develop both intuitions and algorithms for performing mathematics (Ottmar, Landy, Goldstone, & Weitnauer, 2015; Ottmar, Landy, Weitnauer, & Goldstone, 2015).

One core design commitment of Graspable Math is that students must be able to intuitively see linkages between various components of mathematics. Figure 5 shows a screen shot from the system as a student work through the process of solving for two unknowns, x and y. The screen shots do not adequately show the real-time interactive experience and so we encourage readers to visit the project web page. The most immediate, intuitive linking is from one algebraic expression to the next via spatial transformations of the kind shown in Figure 4. Near the bottom right hand corner of Figure 5, one can see that the user is picking up the -1 (shown in red), in the middle of the process, perhaps, of transforming the expression from y=-1+4x into y=4x - 1 or y+1=4x. The commutativity of addition is being shown effectively by the instantaneous reactions of the system; as the user moves the -1 to the right of 4x, the 4x moves over to give room to -1. If the -1 crosses the equal sign, it transforms immediately into a +1. This shows the user a deep mathematical relation: if some Y is equal to a function of some X, then X is also equal to the inverse of that function applied to Y. Graspable Math foregrounds the spatial foundations of valid algebraic reasoning.

Some teachers resist approaches that include spatial transpositions, viewing them to be illegitimate algebraic transformations. They object, “You shouldn’t teach students that they can just move the 2 of y-2=5 to the right side while changing its sign. Students should go through the axiomatically justified steps of adding 2 to both sides of the equation, yielding y-2+2=5+2, and then simplifying to y=5+2.” To this objection, we respond that the teacher’s preferred solution is <u>one</u> justifiable transformation pathway, but mathematics is rich enough to permit multiple axiomatizations of algebra, and the spatial transformations shown in Figure 4 provide an axiomatization that can also be shown to be formally valid. The traditional axiom of addition states: if two quantities are equal and an equal amount is added to each, they are still equal. The alternative, spatial axiomatization has three noteworthy advantages. First, it is a much more psychologically intuitive axiomatization because it has been designed to be efficiently processed by human perception-action systems. Second, it is more efficient, requiring one instead of three transformations. Given that there is non-negligible “fail rate” (e.g. a student getting the wrong result, or giving up entirely) for each transformation, streamlining algebraic transformations is a valuable enterprise. Finally, the transposition operation makes intuitive the general mathematical pattern $Y = F(X) \equiv F^{-1}(Y) = \mathrm{X}$ that is almost completely hidden in the traditional, additive axiom of addition.

Another kind of linkage that Graspable Math allows users to see and create is between algebraic notation and other representations. Figure 5 shows a line graph corresponding to the two equations on each of its sides, where each equation is expressed as a line. The graph-algebra linkage is dynamic and interactive. As users scroll through different values for the constants in the algebraic notation, they see how those values affect the slope and intercept of the lines. Alternatively, if a user manipulates the slope or intercept of the line within the graph, the yoked symbolic numbers will automatically adjust. This kind of bidirectional linkage allows each representation to contribute, where it shines most, to the user's understanding of the underlying mathematics. Relations between lines are highly salient in the graph format. In particular, the intersection point between the lines is conspicuous, and can provide an impetus for students to try to understand what is unique about that point. A fuller answer to that graph-inspired question is provided by the linked notation. In particular, by substituting what Y is equal to from one equation into the Y term for the other leads to the center equation shown below the graph in Figure 5. This center equation, expressed solely in terms of X because the Ys have been eliminated, can then be solved for X, which provides a symbolic tie-in to the point on the X-axis where the two lines intersect. An important part of our philosophy for Graspable Math is that neither the graph nor the algebraic notation is primary or privileged. Both can provide an improved understanding of the other, and when placed in correspondence, allow an understanding that transcends what either can provide on its own. For example, a student who first solves for X and Y algebraically may find themselves wondering what this solution looks like on the graph. If she plots the point corresponding to the solution of the two-equation system shown in Figure 5, {-2,-9}, then she augments her understanding of this solution to (literally) see it as the single point that lies on both of the lines corresponding to the two equations.

A third kind of linkage, depicted in green in Figure 5, is between the different steps of a derivation. Graspable Math uses interactive animations to show the spatial transformations that connect adjacent steps in a derivation, but often times it is useful for mathematical reasoners to see the overall correspondences between elements of a long derivation. In converting the equation of a line from the point-slope form of $y-7=3(x-2)$ to slope-intercept form of $y=3x+1$, it is illuminating to see why and how the intercept depends on both the point and the slope of the original form, but the slope does not. Graphically speaking, if a line needs to hit a particular point {2,7}, then as the slope of the line becomes increasingly shallow, it will have to hit the Y-intercept at increasingly high points. Algebraically speaking, the intercept is seen to depend on the y-coordinate of the point {2,7} in a directly proportional way, but is seen to depend on the x-coordinate in a multiplicative manner with the slope. Together, these twin lenses onto linear systems take advantage of the superior spatial relation highlighting of the graph representation and the superior highlighting of quantity dependencies of the algebraic representation.

We have engaged in a considerable amount of testing of Graspable Math in classroom and informal learning contexts. Students are generally highly enthusiastic and engaged when interacting with the system. Achieving these high levels of engagement is, in itself, a major achievement in educational design given that most students find algebra to be one of the least liked topics in all of their K-12 school experience! Moreover, on tests of transfer of mathematical reasoning to new problems presented in a paper-and-pencil format, Graspable Math has been shown to have educational benefits compared to standard practices for teaching the same content (Ottmar, Landy, Goldstone, & Weitnauer, 2015).

One lingering concern may be whether students could become dependent on the spatial transformations that Graspable Math foregrounds, overgeneralizing them inappropriately. For instance, when deploying Graspable Math in educational contexts, instructor-coaches should be aware of the potential for students to acquire "mal rules" – procedural transformations that are not formally valid (Sleeman, 1984). For example, because cancelling out terms is so aesthetically and kinesthetically agreeable for many students, there is some tendency to overgeneralize from those situations where spatial cancelling is valid (e.g. crossing out the 3s in 3x/3y) to those where it is not (e.g. (3+x)/(3+y)). One might draw the conclusion from these infelicitous perception-action generalizations that these seductively intuitive but dangerous perception-action routines should be avoided. Relying on formal and explicit rules may be safer (Kirshner & Awtry, 2004). However, we have had success by instead training perception-action routines to become more nuanced in their triggering conditions and deployment. One way to train this subtlety is by placing the forms 3x/3y and (3+x)/(3+y) side by side, showing that the cancelling transformation can apply to one but not other, and then showing what the allowed spatial transformations of the latter form is: (3/(3+y)) + (x/(3+y)). The side by side juxtaposition of situations in which canceling 3s is and is not valid is a powerful pedagogical strategy for highlighting critical differences between the forms (Gentner, Loewenstein, & Hung, 2007; Weitnauer, Carvalho, Goldstone, & Ritter, 2014).

## RECOMMENDATIONS AND FUTURE RESEARCH

When thinking about a content domain, there is a tendency to focus on the facts, explicit strategies, and formal models that underlie the domain. While these are certainly important components, it is easy to neglect the more implicit "feels" that an expert develops for a domain (Goodwin, 1994). These feels are often underappreciated precisely because they are implicit and hard to put into words. Even for the apparently abstract and formal content domain of algebra, proficient practitioners develop feels that have a surprisingly large impact on what they are able to accomplish. These feels develop at both input and output ends of information processing. On the input end, mathematicians develop strong dispositions to perceptually organize and re-organize mathematical objects in ways that help them see patterns that are important to them. On the output end, mathematicians develop action routines that help them transform math in revealing ways. Moreover, the input and output sides are inextricably linked because what we perceive influences the actions we generate, and our actions transform the world to make fruitful perceptions more likely (Landy & Goldstone, 2007). Indeed, the most transformative cultural artifacts are those that, like algebraic notation, transform complex conceptual problems into simple perceptual and spatial tasks—the kinds of tasks that we, as a species, have evolved to perform rapidly and reliably (Hutchins, 1995).

Given this perspective, our foremost recommendation going forward is to consider ways to incorporate perception-action procedures into models of domain knowledge. For education purposes, it is beneficial for coaches, trainers, teachers, and students to think about ways of adapting how to see and how to act. Although this kind of knowledge can be considered to be procedural, it is quite different from traditional procedural knowledge that is modeled after following recipes, rules or algorithms. In algebra, it is one thing to explicitly know that the distributive property of multiplication over addition can be used to transform 3(X+Y) into 3X+3Y and another thing to reliably enact the spatial action routine that splits the 3 into the proper number of terms within the parenthesis and constructs the formally valid written expression. Perception and action procedures will likely involve different effective training techniques than knowledge that involves either declarative knowledge or recipe following (Koedinger, Booth, & Klahr, 2013). Compared to more explicit knowledge, embodied and grounded perception-action routines seem to benefit from prolonged and spaced practice, tight agent-to-environment coupling, scaffolded training support, and training that emphasizes active construction over passive study of solved problems.

Although explicit knowledge and perception-action procedures should be distinguished for purposes of optimizing their acquisition and use, it would be a mistake to treat the two kinds of knowledge as operating independently, in parallel. In fact, our second recommendation is to understand understanding itself as the interplay between explicit knowledge and implicit perception-action training. It may be tempting to try tackling the tasks of training explicit and implicit knowledge separately. Much of modern neuroimaging encourages treating different brain regions as modules whose activity levels can be assessed separately. This leads naturally to an approach toward training that stresses the importance of increasing or decreasing the activity of particular modules, much as one would use bicep curls in weightlifting to strengthen the biceps in a targeted, muscle-specific way. We propose an alternative vision: successful training will involve the coordination of modules rather than their independent strengthening (Schwartz & Goldstone, 2016). Proficient mathematicians think strategically about ways of training over time their perception-action systems to do the Right Thing, formally speaking. They also think creativity and laboriously about ways to coordinate between algebraic, geometric, topological, spatial, definitional, and model-based understandings of a situation. If our goal is to promote learning outcomes that are efficient, robust, and broadly applicable, then there are great potential payoffs to developing learning contexts that allow our explicit and implicit understandings to mutually inform one another.

## REFERENCES


Anderson, J. R. (2007). *How can the human mind exist in the physical world?* Oxford, England: Oxford University Press.

Braithwaite, D. W., Goldstone, R. L., van der Maas, H. L . J., & Landy, D. H. (2016). Informal mechanisms in mathematical cognitive development: The case of arithmetic. *Cognition, 149*, 40-55.

Cajori, F. (1928). *A history of mathematical notations*. Open Court Publishing Company: La Salle, Illinois.

Carey, S. (2009). *The Origin of Concepts.* New York: Oxford University Press.

Gentner, D., Loewenstein, J., & Hung, B. (2007). Comparison Facilitates Children's Learning of Names for Parts. *Journal of Cognition and Development, 8*, 285-307.

Goldstone, R. L., & Barsalou, L. (1998). Reuniting perception and conception. *Cognition, 65,* 231-262.

Goldstone, R. L., de Leeuw, J. R., & Landy, D. H. (2015). Fitting Perception in and to Cognition. *Cognition*, *135*, 24-29.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science, 2*, 265-284.

Goodwin, C. (1994). Professional Vision. *American Anthropologist* 96(3): 606-633.

Hutchins, E. (1995). *Cognition in the Wild*. MIT press.

Kellman, P. J., Massey, C. M & Son, J. (2010). Perceptual learning modules in mathematics: Enhancing students' pattern recognition, structure extraction, and fluency. *Topics in Cognitive Science*, *2*, 285-305.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, *35* (4), 224-257.

Koedinger, K., Booth, J., & Klahr, D. (2013). Instructional complexity and the science to constrain it. *Science*, *342*, 935-937.

Landy, D., Allen, C. & Zednik, C. (2014). A perceptual account of symbolic reasoning. *Frontiers in Psychology, 5*, 275. doi: 10.3389/fpsyg.2014.00275

Landy, D., & Goldstone, R. L. (2007a). How abstract is symbolic thought? *Journal of Experimental Psychology: Learning, Memory, & Cognition, 33*, 720-733.

Landy, D., & Goldstone, R. L. (2007). Formal notations are diagrams: Evidence from a production task. *Memory & Cognition, 35*, 2033-2040.

Landy, D. H., Jones, M. N., & Goldstone, R. L. (2008). How the appearance of an operator affects its formal precedence. *Proceedings of the*

*Thirtieth Annual Conference of the Cognitive Science Society,* (pp. 2109-2114). Washington, D.C.: Cognitive Science Society.

Landy, D. H., & Goldstone, R. L. (2010). Proximity and precedence in arithmetic. *Quarterly Journal of Experimental Psychology, 63*, 1953-1968.

Newell, A., and Simon, H.A. (1976). Computer science as empirical enquiry: Symbols and search. *Communications of the ACM 19*(3), 113-126.

Ottmar, E.R., Landy, D., Goldstone, R., & Weitnauer, E. (2015). Getting From Here to There!: Testing the Effectiveness of an Interactive Mathematics Intervention Embedding Perceptual Learning. *Proceedings of the 37th Annual Conference of the Cognitive Science Society*. Pasadena, California: Cognitive Science Society.

Ottmar, E.R., Landy, D., Weitnauer, E. & Goldstone, R. (2015). Graspable Mathematics: Using Perceptual Learning Technology to Discover Algebraic Notation. in Maria Meletiou-Mavrotheris, Katerina Mavrou, and Efi Paparistodemou, eds., *Integrating Touch-Enabled and Mobile Devices into Contemporary Mathematics Education.* Hershey, PA: IGI Global.

Palmer, S., & Rock, I. (1994). Rethinking perceptual organization: The role of uniform connectedness. Psychonomic Bulletin & Review, 1, 29-55.

Quine, W. V. (1977). Natural kinds. In S. P. Schwartz, ed., *Naming, necessity, and natural kinds*. Ithaca, NY: Cornell University Press.

Schwartz, D. L, & Goldstone, R. L. (2016). Learning as coordination: Cognitive psychology and education. In L. Corno & E. M. Anderman (Eds.) *Handbook of Educational Psychology, 3rd edition.* New York: Routledge (pp. 61-75).

Sleeman, D. (1984). An attempt to understand students' understanding of basic algebra, *Cognitive Science*, *6*, 387-412.

Ullman, S. (1984). Visual routines. *Cognition*, *18*, 97-159.

Vygotsky, L. S. (1962). *Thought and language*. Cambridge, MA: MIT Press.

Weitnauer, E., Carvalho, P. F., Goldstone, R. L., & Ritter, H. (2014). Similarity-based ordering of instances for efficient concept learning. *Proceedings of the Thirty-Sixth*

*Annual Conference of the Cognitive Science Society*. (pp. 1760-1765). Quebec City, Canada: Cognitive Science Society.

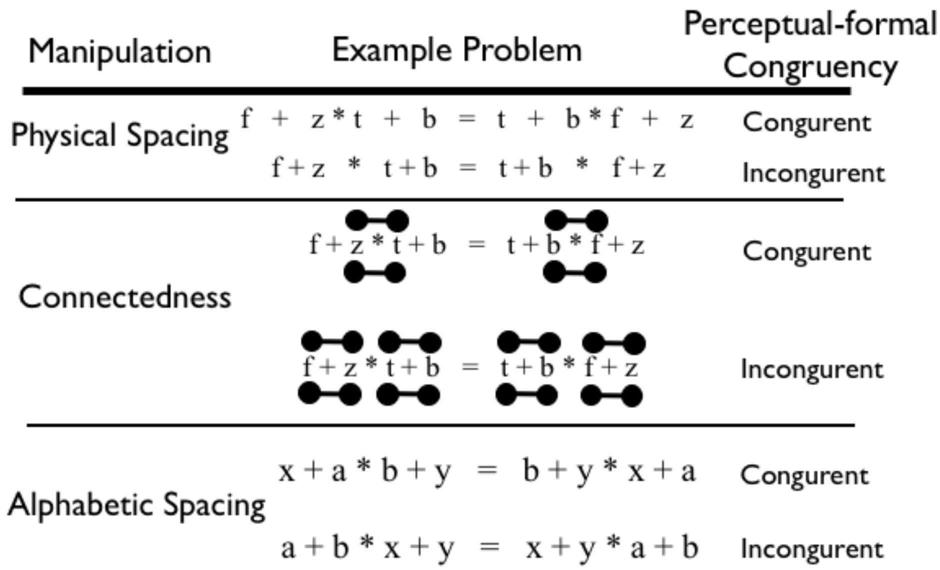


Figure 1. Samples from three experiments reported by Landy and Goldstone (2007). Participants were asked to verify whether an equation is necessarily true. Grouping suggested by factors such as physical spacing, connectedness of contextual geometric forms, and proximity in the alphabet, could be either congruent or incongruent with the order of precedence of arithmetical operators (e.g. multiplications are calculated before additions). The physical manipulations shown all bias participants to perceptually group the symbolic expressions. When formed perceptual groups are congruent with formal order of precedence then validity judgments are much more accurate than when they are incongruent.

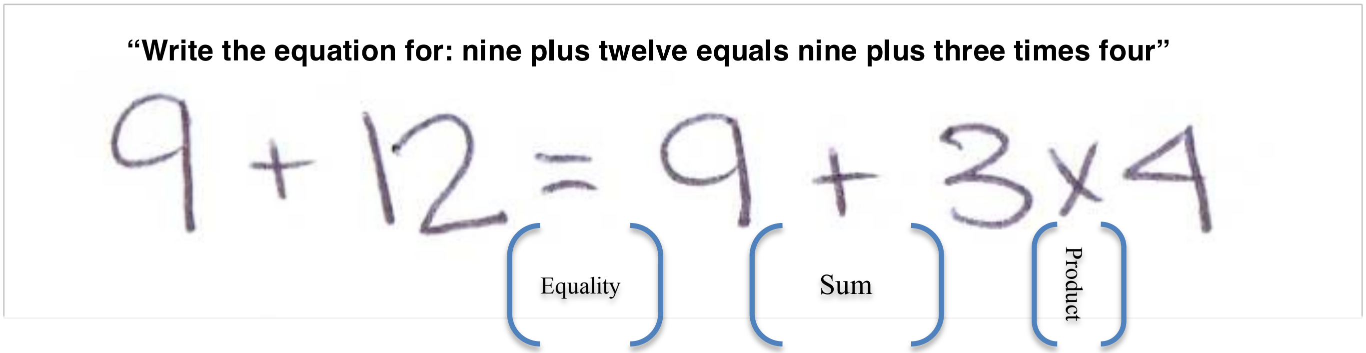


Figure 2. An example of a participants' drawn symbolic representation of an equation expressed in an English sentence, taken from Landy and Goldstone (2010). The physical spacing around the "=", "+", and "X" was measured and compared. Notice how in the drawn equation, the widest spacing is found around the "=" and "+" sign, while the spacing around the "x" is much smaller.

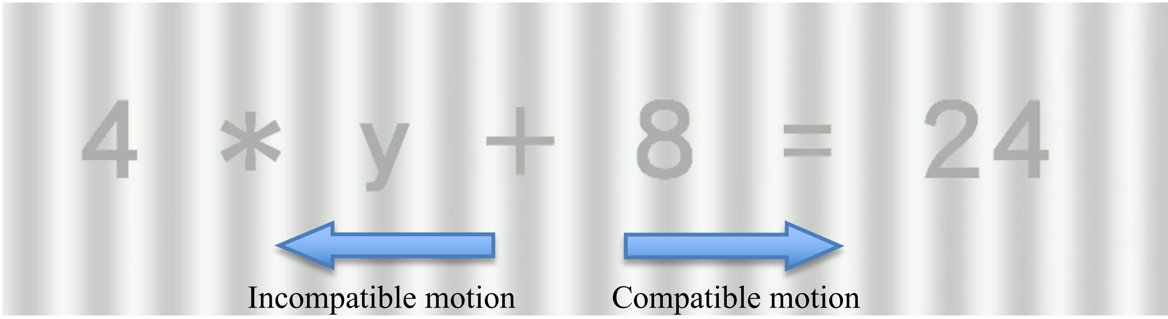


Figure 3. As participants solved for the variable in equations like the above, a vertically oriented grating continuously moved either to the left or to the right. Although irrelevant for the task, when the movement of the grating was compatible with the movements of the numbers required by spatial transposition, participants were more accurate.

| Spatial Transformation | Initial State | Transformed State |
| --- | --- | --- |
| Swapping | $3 = (x - 6)\times(7 + y)$ | $3 = (7 + y)\times(x - 6)$ |
| Splitting | $y = a\times(5 + 7)$ | $y = a\times(5a + 7a)$ |
| Transposing | $x + 3 = 8$ | $x + 3 = 8 - 3$ |
| Simplifying | $y = 3 + 4 \times 7$ | $y = 3 + \overset{28}{4 \times 7}$ |
| Cancelling | $z = \frac{3x}{3y}$ | $z = \frac{\cancel{3}x}{\cancel{3}y}$ |

Figure 4. Examples of physical transformations within notational space. Operations like swapping factors, splitting a variable to make identical copies, transposing a term from one side of an equation to the other, simplifying by replacing one expression with another, and canceling factors in a numerator and denominator are commonly observed in mathematical reasoners, and are often employed in a cognitively efficient and valid fashion.

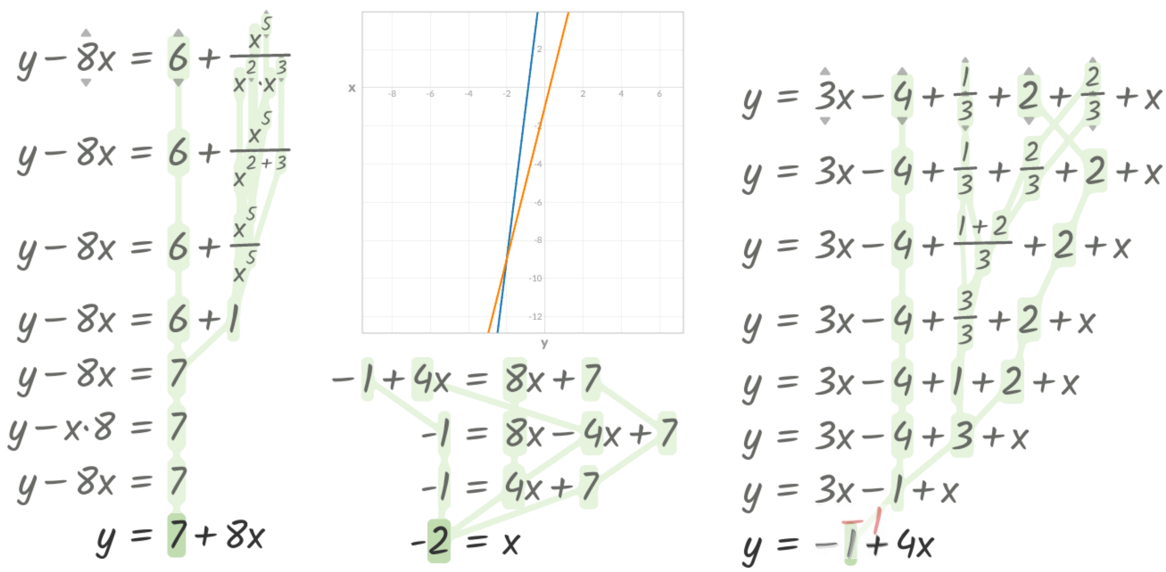


Figure 5. A screen shot from Graspable Math (http://graspablemath.com). Three kinds of linkages between representations are shown. First, interactive animations, such as the moving red - 1 in the lower right corner, provide intuitive, dynamic, and spatial linkages between successive steps in an algebraic derivation. Second, the green pipelines show how symbolic elements are related to each other across several steps of a derivation. Third, the linear equations on the left and right sides are dynamically linked to the graph in the middle. Manipulating the graph's lines instantaneously affects the symbolic quantities in the equations, and vice versa.