Systematic Risk*

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Abstract

We rely on the Aumann-Shapley solution concept to offer a general measure of systematic risk, capturing the contribution of an asset to the risk of a portfolio. Our measure applies to a wide class of risk measures, potentially accounting for high distribution moments, rare disasters, and other risk attributes. In the special case where risk is measured using “variance,” our measure coincides with the traditional “beta.” We then study a general equilibrium setting in which investors trade off expected return for risk, where the term “risk” is broadly defined. We provide sufficient conditions for two-fund money separation, and for the efficiency of the market portfolio. Finally, we derive a general version of the security market line in which our new measure of systematic risk emerges naturally as a generalization of “beta.”

1 Introduction

During the recent crisis in financial markets we have witnessed both the market and individual assets being hit by catastrophic events whose ex-ante probabilities were considered negligible. These events demonstrate that risk is a complex concept, much broader than what is measured by the variance of the returns of an asset. Given this, one would expect measures of systematic risk to capture the contribution of an asset to the different facets of risk.

In this paper we propose a general measure of systematic risk that applies to a broad class of risk measures, potentially accounting for high distribution moments, rare disasters, as well as other aspects of risk. Our measure of systematic risk captures the marginal contribution of an asset to the risk of a portfolio. We then study an equilibrium setting in an asset market where investors trade off expected return against risk, where risk is a quite general concept. We derive sufficient conditions for two-fund money separation, and establish a generalized version of the Capital Asset Pricing Model (Sharpe (1964), Lintner (1965a,b), and Mossin (1966)), in which our

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measure of systematic risk emerges naturally as a generalization of “beta” in the
security market line.

We begin with a broad definition of what would constitute a measure of risk. We
deﬁne a risk measure as any mapping from random variables to real numbers. That
is, a risk measure is simply a summary statistic that encapsulates the randomness
using just one number. The variance (or standard deviation) is obviously the most
commonly used risk measure. However, many other risk measures have been proposed
and used. For example, high distribution moments can account for skewness and tail
risk; downside risk accounts for the variation in losses; and value-at-risk is a popular
measure of disaster risk. Recently, Aumann and Serrano (2008) and Foster and Hart
(2009) offered two appealing risk measures that account for all distribution moments
and for disaster risk. All of these measures fall under our wide umbrella of risk
measures. Moreover, any linear combination of risk measures is itself a risk measure.
Thus, one can easily create measures of risk that account for a number of dimensions
of riskiness, assigning the required weight to each dimension.

In many cases of interest one would like to estimate the contribution of one asset
to the risk of a portfolio of assets. For example, the asset pricing literature has tied
the return of a ﬁnancial asset to the return of the market through “beta,” which is a
measure of the contribution of the asset to the variance of the market portfolio. Sim-
ilarly, it may be desirable to estimate the contribution of banks and other ﬁnancial
institutions to the total market risk (known as systemic risk). To come up with a gen-
eral enough deﬁnition of systematic risk we borrow from the cooperative game theory
literature by relying on the Aumann-Shapley solution concept (Aumann and Shapley
(1974)), which is a continuous generalization of the Shapley value (Shapley (1953)).
This solution concept provides a measure of the contribution of an inﬁnitesimal player
to the surplus obtained by a coalition of players.

The approach we take in this paper is to view a portfolio as a “grand coalition” of
assets. Intuitively, one can think of this grand coalition as a “heap” of assets, where
the amount of each asset in the heap is determined by the dollar amount invested in
the asset. Coalitions in this setting are just subsets of the heap, representing different
ways to construct the portfolio from individual assets. To measure the systematic
risk of a particular asset relative to a portfolio, we need to calculate the average con-
tribution of the asset to the risk of the portfolio across all possible ways to construct
this portfolio. Thus, the measure needs to “integrate” through all possible ways to
construct the portfolio, and calculate the average contribution of adding an inﬁnites-
imal unit of an asset to the total risk. This calculation seems daunting due to the
divisible nature of ﬁnancial assets. However, Aumann and Shapley came up with
an ingenious diagonal formula that signiﬁcantly simpliﬁes this process. Essentially,
they show that instead of integrating over all possible ways to construct the “grand
coalition” of assets, one can restrict attention to a small and simple subset of combi-
nations of assets that forms a “perfect sample” of all possible ways to construct this
grand coalition. We thus deﬁne the systematic risk of an asset relative to a portfolio

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1See Hart (2011) for a uniﬁed treatment of these two measures and Kadan and Liu (2012) for an
analysis of the moment properties of these measures.
as a scaled version of the Aumann-Shapley solution, where the scaling is introduced to obtain a unit-free measure.

In general, our systematic risk measure may be a function of the dollar amount invested in each asset in the portfolio. However, when the risk measure is homogeneous of some degree (as is often the case), dollar amounts become irrelevant, and the systematic risk measure depends on portfolio weights only. Moreover, when the risk measure is homogeneous, our systematic risk measure takes a particularly simple form. The measure is easy to calculate for a variety of risk measures using simple calculus methods. Importantly, when risk is measured using “variance,” our measure boils down to the familiar beta. More generally, we show how our systematic risk measure can be calculated in closed form for a variety of risk measures accounting for high distribution moments and rare disasters. Thus, we obtain a natural and tractable method to calculate the contribution of an asset to the risk of a portfolio of assets, where risk is broadly defined.

Our next step is to show that our measure of systematic risk arises naturally in an equilibrium model generalizing the Capital Asset Pricing Model (CAPM) to a broad set of risk measures. The idea is simple. In the classic CAPM setting investors are assumed to have mean-variance preferences. That is, their utility is increasing in the expected payoff and decreasing in the variance of their payoffs. In our generalized setting we assume that investors have mean-risk preferences, where the term “risk” stands for a host of potential risk measures.

We consider an exchange economy with a finite number of risky assets, one risk-free asset, and a finite number of investors with mean-risk preferences. As usual, in equilibrium each investor chooses a portfolio of assets from the set of efficient portfolios, minimizing risk for a given expected return. However, due to the generality of the risk measure, the geometry of this set is more complicated than in the case where risk is measured by the variance. Nevertheless, we establish sufficient conditions under which the solution to each investor’s problem satisfies Tobin’s (1958) two-fund money separation property. That is, each investor’s optimal portfolio of assets can be presented as a linear combination of the risk-free asset and a unique portfolio of risky assets. The sufficient conditions we propose consist of the following four properties of risk measures:

- **Convexity**: The risk of the portfolio of assets (with a positive weight assigned to each asset) is smaller than the corresponding weighted average risk of the constituent assets.

- **Homogeneity**: Scaling up a random investment by a factor $t > 0$ increases risk by a factor of $t^k$ for some $k$.

- **Translation invariance**: Adding the same scalar to two random variables changes their riskiness in a manner that preserves the risk ordering of these two variables.

- **Risk-free property**: Assets paying a constant amount are the least risky assets, and adding a risk-free asset (paying a positive amount) to a risky asset reduces total risk.
We demonstrate that a large class of risk measures satisfy these sufficient conditions, where the variance is just one special case. A consequence of two-fund money separation is that the equilibrium market portfolio lies on the efficient frontier. Using this, and under a smoothness condition we establish a generalization of the classic security market line (SML) to a large class of risk measures. Specifically, in equilibrium, the expected return of each risky asset $i$ satisfies

$$E(z_i) = r_f + B_i (E(z^M) - r_f),$$

where $z_i$ is the risky return of asset $i$, $z^M$ is the risky return of the market portfolio, $r_f$ is the risk-free rate, and $B_i$ is our measure of systematic risk, based on the Aumann-Shapley solution concept. Thus, our measure of systematic risk emerges naturally in an equilibrium setting as a generalization of beta. In particular, in equilibrium, each asset’s expected return reflects its average marginal contribution to the total risk (broadly defined) of the market portfolio, across all possible ways to construct the market portfolio.

Our paper contributes to several strands of the literature. First, the paper adds to the growing literature on risk measurement. This literature dates back to Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970) who extend the notion of riskiness beyond the “variance” framework by introducing stochastic dominance rules. Artzner, Delbaen, Eber, and Heath (1999) specify desirable properties of coherent risk measures. More recently Aumann and Serrano (2008), Foster and Hart (2009, 2011), and Hart (2011) came up with appealing risk measures that generalize conventional stochastic-dominance rules. Notably, all the risk measures discussed in this literature are idiosyncratic in nature. Our paper adds to this literature by specifying a method to calculate the systematic risk of an asset for any given risk measure. This in turn allows us to study the fundamental risk-return trade-off associated with a risk measure.

Our paper also adds to the recent literature on the measurement of systemic risk, which is the risk that the entire economic system collapses. Adrian and Brunnermeier (2008) define the $\Delta CoVaR$ measure as the difference between the value-at-risk of the banking system conditional on the distress of a particular bank and the value-at-risk of the banking system given that the bank is solvent. Acharya, Pedersen, Philippon, and Richardson (2010) propose the Systemic Expected Shortfall measure (SES), which estimates the exposure of a particular bank in terms of under-capitalization to a systemic crisis. Our paper takes a general approach to the problem of estimating the contribution of one asset to the risk of a portfolio of assets. We rely on an established game theoretical solution to provide an easy-to-calculate and intuitive measure that applies to a wide variety of risk measures, as well as in an array of contexts.

The paper also contributes to the growing literature on high distribution moments and disaster risk and their effect on prices. Kraus and Litzenberger (1976), Jean (1971), Kane (1982), and Harvey and Siddique (2000)) argue that investors favor right-skewness of returns, and demonstrate the cross-sectional implications of this effect. In addition, Barro (2006, 2009), Gabaix (2008, 2012), Gourio (2012), Chen, Joslin, and Tran (2012), and Wachter (2012)) study the aversion of investors to tail
risk and rare disasters. Our paper adds to this literature by outlining a general measure of systematic risk that can capture the contribution of an asset to a range of market risk aspects such as high distribution moments, rare disasters, and downside risk. Our measure applies to homogeneous and non-homogeneous risk measures, and can be calculated easily when one needs to estimate the contribution of a particular asset to the risk of a portfolio. Additionally, under some additional assumptions, we show that our measure emerges naturally in an equilibrium setting. Our approach to equilibrium essentially follows a reduced form, where preferences are described through the aversion to broadly defined risk. It should be emphasized, however, that our approach is stylized and does not account for the dynamics of returns, cash flows and consumption as do modern consumption based asset pricing models (e.g., Bansal and Yaorn (2004) and Campbell and Cochrane (1999)). These models rely on the specification of a utility function (such as Epstein and Zin (1989) preferences or preferences reflecting past habits). One advantage of our approach is that it provides a very parsimonious and simple one factor model that can capture different aspects of risk in a manner that may lend itself naturally to empirical investigation.

Finally, the paper adds to an extensive list of papers applying the Aumann-Shapley solution concept to allocate costs in different contexts. For example, Billera, Heath, and Raanan (1978) solve a telephone billing allocation problem, Samet, Talmor, and Zang (1984) solve a transportation costs allocation problem, and Powers (2007) studies the allocation of insurance risk. Billera, Heath, and Verrecchia (1981) use a related procedure to allocate production costs. Our approach to defining systematic risk is similar in spirit as we essentially allocate total risk among the individual assets of a portfolio.

The paper proceeds as follows. In Section 2 we define the notion of risk measures and study their different properties. Section 3 introduces the Aumann-Shapley solution concept and its application to the definition of systematic risk. In Section 4 we study the equilibrium setup and offer a generalization of the CAPM. Section 5 concludes. Technical proofs are in Appendix I.

2 Risk Measures and Their Properties

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where $\Omega$ is the state space, $\mathcal{F}$ is the $\sigma$-algebra of events, and $P(\cdot)$ is a probability measure. As usual, a random variable is a measurable function from $\Omega$ to the reals. In the context of investments, we typically consider random variables representing the payoffs or the returns of financial assets. Thus, we often refer to random variables as “investments” or “random returns.” We generically denote random variables by $\tilde{z}$, which is a shorthanded notation for $\tilde{z}(\omega)$ $\forall \omega \in \Omega$. We restrict attention to random variables for which all moments exist. We denote the expected value of $\tilde{z}$ by $E(\tilde{z})$ and its $k^{th}$ central moment by $m_k(\tilde{z})$, where $k \geq 2$.

A risk measure is simply a function that assigns to each random variable a single number summarizing its riskiness. Formally,

**Definition 1** A risk measure is a function mapping random variables to the reals.
We generically denote risk measures by $R(\cdot)$. The simplest and most commonly used risk measure is the the variance ($R(\tilde{z}) = \sigma^2(\tilde{z})$). However, many other risk measures have been proposed in the literature, capturing higher distribution moments. Risk measures can be applied to individual random variables or to portfolios of random variables. Formally, assume there are $n$ random variables represented by the vector $\tilde{z} = (\tilde{z}_1, ..., \tilde{z}_n)$. A portfolio is represented by a vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ where $x_i$ is the dollar amount invested in $\tilde{z}_i$. Then, $\mathbf{x} \cdot \tilde{z} = \sum_{i=1}^{n} x_i \tilde{z}_i$ is itself a random variable. We then say that the risk of portfolio $\mathbf{x}$ is simply $R(\mathbf{x} \cdot \tilde{z})$. When the vector of random variables is clear, we often abuse notation and denote $R(\mathbf{x})$ as a shorthand for $R(\mathbf{x} \cdot \tilde{z})$.

We next discuss several properties of risk measures that will become useful later, and demonstrate them using some popular risk measures.

### 2.1 Properties of Risk Measures

The first common property of risk measures is convexity. Formally, we say that a risk measures $R$ is convex if for any two random returns $\tilde{z}_1$ and $\tilde{z}_2$, and for any $\lambda \in (0, 1)$, we have

$$R(\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2) \leq \lambda R(\tilde{z}_1) + (1 - \lambda) R(\tilde{z}_2),$$

with equality holding only when $\tilde{z}_1 = \tilde{z}_2$ with probability 1. Notice that $\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2$ can be considered as the return of a portfolio that assigns weights $\lambda$ and $1 - \lambda$ to $\tilde{z}_1$ and $\tilde{z}_2$, respectively. Then the convexity condition says that the risk of the portfolio should not be higher than the corresponding weighted average risk of the constituent investments. Thus, convexity of a risk measure captures the idea that diversifying among two investments lowers the total risk.

The next property is homogeneity. A risk measure $R(\cdot)$ is homogeneous of degree $k$, if for any random return $\tilde{z}$ and positive number $\lambda > 0$,

$$R(\lambda \tilde{z}) = \lambda^k R(\tilde{z}).$$

This condition guarantees, among other things, that the risk ranking between two investments does not depend on scaling.

Next we consider the issue of how adding a scalar to a random return affects its riskiness. We say that a risk measure $R(\cdot)$ is translation invariant if for any two random returns $\tilde{z}_1$ and $\tilde{z}_2$, and for any real number $\lambda \in \mathbb{R}$, $R(\tilde{z}_1) \leq R(\tilde{z}_2)$ if and only if $R(\tilde{z}_1 + \lambda) \leq R(\tilde{z}_2 + \lambda)$.

Intuitively, this condition implies that adding a constant does not change the risk ranking of two investments. In other words, risk measures which are translation invariant are unaffected by adding a constant investment.

Next we would like to formalize a property dealing with the type of assets that are risk-free. If an asset $\tilde{z}$ has $R(\tilde{z}) = 0$ we say that $\tilde{z}$ is $R$-risk-free. We say that a risk measure $R(\cdot)$ has the risk-free property, if (i) $R(\tilde{z}) \geq 0$ for all $\tilde{z}$; (ii) $\tilde{z}$ is $R$-risk-free if and only if $P(\{\tilde{z} = c\}) = 1$ for some constant $c$; and (iii) $R(\tilde{z}_1 + \tilde{z}_2) \leq R(\tilde{z}_1)$ whenever $\tilde{z}_2$ is $R$-risk-free with $P(\{\tilde{z}_2 > 0\}) = 1$. Namely, $R$ has the risk-free property whenever $\tilde{z}_2$ is $R$-risk-free with $P(\{\tilde{z}_2 > 0\}) = 1$. Namely, $R$ has the risk-free property whenever $\tilde{z}_2$ is $R$-risk-free with $P(\{\tilde{z}_2 > 0\}) = 1$.

The term “translation invariance” is similar but more general than in Artzner et al. (1999), who define $R$ to be translation invariant if $R(\tilde{r} + \lambda) = R(\tilde{r}) - \lambda$.  

\[\footnote{The term “translation invariance” is similar but more general than in Artzner et al. (1999), who define $R$ to be translation invariant if $R(\tilde{r} + \lambda) = R(\tilde{r}) - \lambda$.}\]
if the only risk-free assets are those that pay a constant amount with probability 1, if all other assets have a strictly positive risk, and if adding a positive risk-free asset can only reduce risk.

When we deal with portfolios we often ask that the risk of the portfolio be a smooth function of the amount invested in each asset. Formally, we say that a risk measure is smooth if for any vector of random returns \( \tilde{z} = (\tilde{z}_1, ..., \tilde{z}_n) \) and for all portfolios \( x = (x_1, ..., x_n) \) we have that \( R(x \cdot \tilde{z}) \) is continuously differentiable in \( x_i \) for \( i = 1, ..., n \).

Finally, note that all of the properties we discussed are maintained when taking convex combinations of different risk measures with the same degree of homogeneity. Thus, we can easily create new risk measures satisfying the properties from existing risk measures. That is, let \( s \) be a positive integer and let \( R^1(\cdot), ..., R^s(\cdot) \) be risk measures and choose \( \theta = (\theta_1, ..., \theta_s) \) such that \( \sum_{i=1}^{s} \theta_i = 1 \) and \( \theta_i \geq 0 \) \( \forall i \). We can then define a new risk measure by

\[
R^\theta(\tilde{z}) = \sum_{i=1}^{s} \theta_i R^i(\tilde{z}).
\]

We then have the following trivial but extremely useful lemma.

**Lemma 1** Fix \( k \), assume that each \( R^i \) is convex, homogeneous of degree \( k \), translation invariant, smooth, and satisfies the risk-free property. Then, \( R^\theta \) also satisfies all of these properties.

### 2.2 Examples of Risk Measures

In this section, we examine as examples a number of risk measures and discuss whether they satisfy (or fail) the above properties.

#### 2.2.1 Even Central Moments and Their Variations

The first and most widely used risk measure is the variance, which satisfies all five properties defined above. In fact, all even moments of a random return can be considered as risk measures, and they all satisfy these properties.

**Proposition 1** For all \( k \) even, \( R(\tilde{z}) = m_k(\tilde{z}) \) is a risk measure which is convex, homogeneous of degree \( k \), translation invariant, smooth, and has the risk-free property.

Since the composition of a convex and increasing function with a convex function is convex, we have \( R(\tilde{z}) = (m_k(\tilde{z}))^t \) for \( t \geq 1 \) is also a risk measure that satisfies all the properties. More interesting is that we can take powers that are lower than 1 of even moments and still maintain all of the properties (including convexity, despite the non-convex transformation of taking a root). Specifically, let \( w_k(\tilde{z}) = (m_k(\tilde{z}))^{\frac{1}{k}} \) be the normalized \( k^{th} \) moment. Then,
Proposition 2 For all \( k \) even, \( R(\tilde{z}) = w_k(\tilde{z}) \) is a risk measure which is convex, homogeneous of degree 1, translation invariant, smooth, and has the risk-free property.

Similarly, we can define \( R(\tilde{z}) = (w_k(\tilde{z}))^t \) for any \( t \geq 1 \), and obtain a risk measure that satisfies all of the properties. Finally, using Lemma 1 we can take convex combinations of different even moments by normalizing them to have the same level of homogeneity and relying on Proposition 2 to ensure that convexity is maintained. This creates new risk measures which account for several distribution moments. For example, suppose we would like a risk measure that accounts for both dispersion and tail risk. Such a risk measure should reflect both the second and the fourth central moments. In particular, for all \( k \in [0, 1] \) let

\[
R^\theta(\tilde{z}) = \theta m_2(\tilde{z}) + (1 - \theta) \sqrt{m_4(\tilde{z})}
\]

Then, \( R^\theta(\tilde{z}) \) is a family of risk measures parametrized by \( \theta \), which incorporate both the dispersion (as reflected by the variance \( m_2(\tilde{z}) \)) and the tail risk (as reflected by the fourth central moment \( m_4(\tilde{z}) \)) of a random return. The parameter \( \theta \) specifies the weight assigned to the two moments. A higher \( \theta \) reflects a larger weight assigned to the variance relative to the fourth moment. By Lemma 1 these risk measures satisfy all of the properties, with homogeneity of degree 2.

2.2.2 Odd Central Moments and Their Variations

Odd central moments of a random return, that is, \( R(\tilde{z}) = m_k(\tilde{z}) \) for odd \( k > 2 \), can also serve as risk measures. In contrast to the even central moments, however, neither convexity nor the risk-free property holds in this case. To see the former, consider the simple example of two random returns, \( \tilde{z}_1 \) and \( \tilde{z}_2 \) that are independent and have negative third central moments. Then by independence and homogeneity,

\[
m_3 \left( \frac{1}{2} \tilde{z}_1 + \frac{1}{2} \tilde{z}_2 \right) = \frac{1}{2} m_3(\tilde{z}_1) + \frac{1}{2} m_3(\tilde{z}_2) > \frac{1}{2} m_3(\tilde{z}_1) + \frac{1}{2} m_3(\tilde{z}_2),
\]

since \( m_3(\tilde{z}_1) + m_3(\tilde{z}_2) < 0 \). To see the latter, note that the third moment can be negative, violating the risk-free property. Nevertheless, the odd central moments satisfy the rest of the properties, as stated in the next proposition.

Proposition 3 For all \( k > 2 \) odd, \( R(\tilde{z}) = m_k(\tilde{z}) \) is a risk measure which is homogeneous of degree \( k \), translation invariant, and smooth.

Similar to the even central moments, we can define \( R(\tilde{z}) = (m_k(\tilde{z}))^t \) for any \( t \geq 1 \), and it is apparent that the resulting risk measure satisfies again homogeneity, translation invariance, and smoothness, but not the other two properties. Then letting \( w_k(\tilde{z}) = (m_k(\tilde{z}))^{\frac{1}{k}} \) be the normalized \( k^{th} \) moment, we have

Proposition 4 For all \( k > 2 \) odd, \( R(\tilde{z}) = w_k(\tilde{z}) \) is a risk measure which is homogeneous of degree 1, translation invariant, and smooth.
2.2.3 Value-at-Risk and Its Variations

Another risk measure widely used in financial risk management is the value-at-risk (VaR) designed to capture the risk associated with rare disasters or downside risk. VaR measures the amount of loss not exceeded with a certain confidence level. Formally, given some confidence level $\alpha \in (0, 1)$, for any random return $\tilde{z}$, the VaR measure $\text{VaR}_\alpha(\tilde{z})$ is defined as the negative of the $\alpha$-quantile of $\tilde{z}$, i.e.,

$$\text{VaR}_\alpha(\tilde{z}) = -\inf \{ z \in \mathbb{R} : F(z) \geq \alpha \},$$

where $F(\cdot)$ is the cumulative distribution function of $\tilde{z}$. Notice that we include the minus sign to reflect the fact that a larger loss indicates higher risk. In particular, when $\tilde{z}$ is continuously distributed with a density function $f(\cdot)$, (1) is implicitly determined by

$$\int_{-\infty}^{-\text{VaR}_\alpha(\tilde{z})} f(z) \, dz = \alpha.$$

Apparently, for any risk-free return $\tilde{z}$ with $P(\{\tilde{z} = c\}) = 1$, we have $\text{VaR}_\alpha(\tilde{z}) = -c$, implying that the VaR of risk-free assets depends on the risk-free return. Hence, the risk-free property is not satisfied. In addition, it is not hard to find examples where convexity is violated for the VaR measure. The rest of the properties are addressed in the following proposition.

**Proposition 5** For any $\alpha \in (0, 1)$, $R(\tilde{z}) = \text{VaR}_\alpha(\tilde{z})$ is homogeneous of degree 1 and translation invariant. If we have a vector of random returns $\tilde{z}$ that follows a joint distribution with a continuously differentiable probability density function, then $R(\tilde{z}) = \text{VaR}_\alpha(\tilde{z})$ is also smooth.

Based on the VaR, we can further construct an expected value-at-risk (ExVaR) measure, which captures the average amount of loss from disastrous events, where a disastrous event is defined as an event involving a loss larger than the VaR. Formally, given some confidence level $\alpha \in (0, 1)$, for any random return $\tilde{z}$, the ExVaR measure $\text{ExVaR}_\alpha(\tilde{z})$ is defined as the negative of the conditional expected value of $\tilde{z}$ below the $\alpha$-quantile. That is,

$$\text{ExVaR}_\alpha(\tilde{z}) = -\frac{1}{\alpha} \int_{-\infty}^{-\text{VaR}_\alpha(\tilde{z})} z F(z) \, dz.$$

(2)

Similar to the VaR, ExVaR does not exhibit the risk-free property. However, the rest of the properties (including convexity) are satisfied. Formally,

**Proposition 6** For any $\alpha \in (0, 1)$, $R(\tilde{z}) = \text{ExVaR}_\alpha(\tilde{z})$ is convex, homogeneous of degree 1, and translation invariant. If we have a vector of random returns $\tilde{z}$ that follows a joint distribution with a continuously differentiable probability density function, $R(\tilde{z}) = \text{ExVaR}_\alpha(\tilde{z})$ is also smooth.
2.2.4 The Aumann-Serrano and Foster-Hart Risk Measures

Two measures of riskiness are recently proposed by Aumann and Serrano (2008, hereafter AS) and Foster and Hart (2009, hereafter FH), which generalize the notion of second order stochastic dominance (SOSD). The AS measure $R_{AS}(\tilde{z})$ is given by the unique positive solution to the implicit equation

$$E\left[\exp\left(-\frac{\tilde{z}}{R_{AS}(\tilde{z})}\right)\right] = 1. \quad (3)$$

The FH measure $R_{FH}(\tilde{z})$ is given by the unique positive solution to the implicit equation

$$E\left[\log\left(1 + \frac{\tilde{z}}{R_{FH}(\tilde{z})}\right)\right] = 0. \quad (4)$$

Hart (2011) shows that $R_{AS}$ and $R_{FH}$ correspond, respectively, to the wealth-uniform dominance order and the utility-uniform dominance order, both of which are complete orders and coincide with SOSD whenever the latter applies. Note that these two measures are only defined for random returns $\tilde{z}$ satisfying $E(\tilde{z}) > 0$ and $P(\{\tilde{z} < 0\}) > 0$. Thus, technically, these measures cannot apply to risk-free assets, and the risk-free property is irrelevant. Moreover, it is not hard to find examples in which translation invariance is violated for both these measures.\(^3\) The rest of the properties are satisfied as stated in the next proposition.

**Proposition 7** Both $R_{AS}$ and $R_{FH}$ are convex, homogeneous of degree 1, and smooth.

3 Systematic Risk and the Aumann-Shapley Value

The systematic risk of an asset is typically conceived as a measure of the contribution of the asset to the risk of a diversified portfolio. The classical approach to this issue uses the variance as a risk measure, in which case systematic risk is measured by the “beta” of the asset. It is the ratio of the covariance of an asset’s return with the portfolio return to the variance of the portfolio return. A priori, it is not clear how to generalize the notion of beta to more general risk measures that account for high distribution moments. In this section we develop a general framework to measure systematic risk that generalizes the traditional beta. To this end, we use concepts from cooperative game theory, which were developed to measure the marginal contribution of individuals to a group. In particular, we rely on the Aumann-Shapley value (Aumann and Shapley (1974)), which is a continuous generalization of the Shapley value (Shapley (1953)). The idea is to view a portfolio as a “coalition” of assets. The systematic risk of an asset will then be the average contribution of this asset to the total risk of the portfolio.

\(^3\)For example, consider two normally distributed random returns $\tilde{z}_1 \sim N(\mu_1, \sigma_1^2)$ and $\tilde{z}_2 \sim N(\mu_2, \sigma_2^2)$. According to Aumann and Serrano (2008), for $i = 1, 2$, $R_{AS}(\tilde{z}_i) = \frac{\sigma_i^2}{\mu_i}$ and $R_{AS}(\tilde{z}_i + \lambda) = \frac{\sigma_i^2}{\mu_i + \lambda}$ for any constant $\lambda$. It is easy to come up with parameter values such that $\frac{\sigma_1^2}{\mu_1} < \frac{\sigma_2^2}{\mu_1}$ but $\frac{\sigma_1^2}{\mu_2 + \lambda} > \frac{\sigma_2^2}{\mu_2 + \lambda}$ or vice versa.
3.1 A Review of the Shapley and Aumann-Shapley Values

Suppose that a group of players (a coalition) can collaborate to guarantee a certain surplus. Since each player may contribute a different amount to the coalition, one would need a way to determine the “fair” payoff to each player. The Shapley value provides one possible solution to this problem. It assigns to each player its average marginal contribution to all possible coalitions to which she can belong.

Formally, a game in coalitional form is a pair \((N, v)\), where \(N = \{1, 2, \ldots, n\}\) denotes a finite set of players, and \(v: \mathcal{P}(N) \rightarrow \mathbb{R}\) is a function that maps subsets of the players to the reals with \(v(\emptyset) = 0.\) A coalition \(S\) is any subset of \(N\), and thus \(v(S)\) can be viewed as the total surplus achieved by coalition \(S\). A value \(\phi\) is then an operator that assigns to each \(v\) a vector \(\phi v = (\phi v(1), \phi v(2), \ldots, \phi v(n))\), where \(\phi v(i)\) represents the contribution of player \(i\). The Shapley value is given by

\[
\phi v(i) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)],
\]

where \(|S|\) denotes the number of players in the coalition \(S\). To see the intuition, suppose that the players arrive according to a random order, and we assume that all orders (i.e., permutations) are equally likely with probability \(\frac{1}{n!}\). If player \(i\) arrives right after coalition \(S\), then her marginal contribution is captured by \(v(S \cup \{i\}) - v(S)\). Notice that there are \(|S|!(n - |S| - 1)!\) different orders (permutations) that give rise to such a situation. Therefore, the Shapley value essentially measures the average marginal contribution of each player across all possible permutations.

The Shapley value is the unique value satisfying the following four axioms (See Shapley (1953)). (i) Efficiency: The total gain is distributed among the players; (ii) Symmetry: If two players have equivalent contributions to all coalitions then their values are equal; (iii) Additivity: The value of a sum of two games is the sum of the values; and (iv) The null-player property: If a player has zero marginal contribution to all coalitions then the value obtained by this player is zero.

The Shapley value is appropriate when dealing with a finite number of players. In our case, we need to deal with portfolios of infinitely divisible financial assets, where we can add an infinitesimally small unit of an asset to the portfolio, and obtain an estimate of the contribution of this small unit to the total risk of the portfolio. The Aumann-Shapley value is a generalization of the Shapley value to a continuum of players that fits this situation.

A formal development of the Aumann-Shapley value can be found in Aumann and Shapley (1974). Here we provide a heuristic (yet quite precise) introduction, based on Mertens (1980). Assume that there is a continuum of players given by an interval \(I \subset \mathbb{R}\). Then, coalitions are all Borel subsets of \(I\), and the surplus obtained by each coalition is given by a function \(v(S)\), assigning a real number to each coalition \(S\). As in the discrete case, a value assigns to each player its average marginal contribution to the surplus of coalitions of players “before her.” The challenge is that given the continuum of players, it is not clear how to order them, and present all permutations.

\(^4\mathcal{P}(N)\) denotes the power-set of \(N\), i.e., the set of all possible coalitions.
A key insight of Aumann and Shapley is that since each individual player here has only a small role, the set \( tI \) for some \( t \in [0, 1] \) representing a proportion \( t \) out of the grand coalition, forms a “perfect sample” of all possible coalitions. Consequently, Aumann and Shapley obtained the following analog to (5):

\[
(\phi v)(ds) = \int_0^1 (v(tI \cup \{ds\}) - v(tI)) \, dt, \tag{6}
\]

where \((\phi v)(ds)\) is the value of an infinitesimal player \( ds \) (corresponding to \( \phi v(i) \) in (5)). The formula (6) is called the “diagonal formula,” since it only considers coalitions formed as proportions of the grand coalition and ignores “permutations” off the diagonal. Of particular interest are games in which \( v(\cdot) \) is a continuously differentiable function of a finite number of non-atomic measures:\(^5\)

\[
v(S) = g(\eta_1(S), \ldots, \eta_n(S)),
\]

where \( g(0, \ldots, 0) = 0. \)

Then, (6) can be written as

\[
\phi v(S) = \sum_{i=1}^n \eta_i(S) \int_0^1 g_i(\eta_1(tI), \ldots, \eta_n(tI)) \, dt, \tag{7}
\]

where \( g_i \) is the \( i^{th} \) partial derivative of \( g \), and \( S \) is any coalition. Aumann and Shapley show that (7) is the unique value satisfying efficiency and symmetry in a class of games where \( v(\cdot) \) is sufficiently smooth.

### 3.2 Aumann-Shapley Based Systematic Risk

When \( v(\cdot) \) captures the surplus obtained by a coalition, the Shapley and the Aumann-Shapley values are often viewed as the entry-fee a player would be willing to pay to participate in the game (assuming payoffs are distributed “fairly”). Analogously, \( v(\cdot) \) is often used to model a cost incurred by a coalition, and the Aumann-Shapley value is then used to study how this cost should be allocated among the different players. In that case, the Aumann-Shapley value is often viewed as a fair penalty that should be imposed on a player due to her contribution to the total cost. We follow this latter route. We model portfolios as coalitions, and use the Aumann-Shapley value to measure the fair contribution of any asset to the total risk of a diversified portfolio, which we define as the asset’s systematic risk. To this end, we follow ideas from Billera, Heath and Raanan (1978) and Samet, Tauman, and Zang (1984), who apply the Aumann-Shapley value to study communication and transportation cost allocation.

Fix a risk measure \( R \) and assume there exist \( n \) risky assets given by the random returns \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n) \). Let \( x = (x_1, \ldots, x_n) \) be the dollar amounts invested in

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\(^5\)A measure \( \eta \) is non-atomic if given any \( S \subset I \) such that \( |\eta(S)| > 0 \), there exists \( T \subset S \) with \( 0 < |\eta(T)| < |\eta(S)| \).
\( z_1, \ldots, z_n \). We allow \( x_i \) to be positive or negative, where a negative amount corresponds to short selling, as long as the total dollar amount \( \bar{x} = \sum_{i=1}^{n} x_i \) is positive. Note that \( x \) is fixed, and that the notion of systematic risk will be defined relative to this particular portfolio of assets. \(^6\) Also, let \( \alpha_i = \frac{x_i}{\bar{x}} \) be the weight assigned to asset \( i \). Then, \( \sum_{i=1}^{n} \alpha_i = 1 \).

We can now use these ingredients to define a game in coalitional form. Formally, let \( I = [0, n) \) be the grand coalition. We view \( I \) as the union of \( I_i = [i-1, i) \) for \( i = 1, \ldots, n \), where \( I_i \) represents the proportion of dollar amounts invested in the \( i^{th} \) asset. A coalition is then a Borel subset of \( I \), capturing the proportion of the total dollar value invested in each asset. For example, the coalition \([0,0.5) \cup [1,1.3)\) corresponds to investing 0.5 dollars in asset 1, 0.3 dollars in asset 2, and zero dollars in all other assets. Thus, any portfolio can be naturally represented by at least one (and typically many) coalitions. We can now define \( n \) non-atomic measures \((\eta_1(\cdot), \ldots, \eta_n(\cdot))\) as follows. For each \( i \), and for each coalition \( S \)

\[
\eta_i(S) = x_i \lambda(S \cap I_i),
\]

where \( \lambda(\cdot) \) is the Lebesgue measure.

We then define the surplus function \( v(\cdot) \) using our risk measure \( R \). In particular, for each coalition \( S \), let

\[
v(S) = R(\eta_1(S), \ldots, \eta_n(S)),
\]

which can be viewed as the risk of investing an amount \( \eta_i(S) \) in the \( i^{th} \) asset for all \( i \). \(^7\) If \( R(0, \ldots, 0) = 0 \) and if \( R(\cdot) \) is smooth then we can apply (7) to obtain the Aumann-Shapley value of any coalition \( S \). In particular,

\[
\phi v^R(S) = \sum_{i=1}^{n} \eta_i(S) \int_{0}^{1} R_i(t \eta_1(I), \ldots, t \eta_n(I)) \, dt
\]

\[= \sum_{i=1}^{n} \eta_i(S) \int_{0}^{1} R_i(tx_1, \ldots, tx_n) \, dt,
\]

where we write \( \phi v^R \) to emphasize the dependence on the choice of the risk measure \( R \). Now, consider a coalition that represents investing one dollar in asset \( i \) and zero in all other assets. We denote such a coalition by \( S_i \). \(^8\) The Aumann-Shapley value of such a coalition corresponds to the contribution of one dollar of asset \( i \) to the risk of the portfolio, and is given in the next proposition.

**Proposition 8** Let \( R(\cdot) \) be a smooth risk measure and suppose that \( R(0, \ldots, 0) = 0 \). Let \( S_i \) be a coalition such that \( \eta_i(S_i) = 1 \) and \( \eta_j(S_i) = 0 \) for \( j \neq i \). Then, the Aumann-Shapley value of \( S_i \) is given by

\[
\phi v^R(S_i) = \int_{0}^{1} R_i(tx_1, \ldots, tx_n) \, dt.
\]

\(^6\)In Section 4, when we study an equilibrium setup, the portfolio \( x \) will arise endogenously as the market portfolio. But here we allow it to be any given portfolio.

\(^7\)Recall that \( R(\eta_1(S), \ldots, \eta_n(S)) \) is a shorthand for \( R((\eta_1(S), \ldots, \eta_n(S)) \cdot \bar{z}) \).

\(^8\)Obviously, \( S_i \) is not unique, but any such coalition would yield the same Aumann-Shapley value.
The units of measurement of $\phi v^R(S_i)$ are “risk per dollar.” In order to obtain a unit-free measure of systematic risk we first divide $\phi v^R(S_i)$ by the total risk of the portfolio and then multiply by total dollar investment. Thus, we define the systematic risk of asset $i$ by

$$B^R_i = \frac{\bar{x} \cdot \phi v^R(S_i)}{R(x_1, ..., x_n)} = \frac{\bar{x} \int_0^1 R_i(tx_1, ..., tx_n) dt}{R(x_1, ..., x_n)}. \tag{10}$$

Intuitively, the systematic risk $B^R_i$ is a normalized measure of the average contribution of asset $i$ to the risk of the portfolio.

In most cases of interest $R(\cdot)$ is homogeneous of some degree $k$, and hence $R_i(\cdot)$ is homogeneous of degree $k - 1$ for all $i$. In this case, (9) can be simplified as

$$\phi v^R(S_i) = R_i(x_1, ..., x_n) \int_0^1 t^{k-1} dt = \frac{R_i(x_1, ..., x_n)}{k}. \tag{11}$$

Thus, when $R$ is homogeneous the systematic risk takes the following form

$$B^R_i = \frac{\bar{x} \cdot R_i(x_1, ..., x_n)}{k \cdot R(x_1, ..., x_n)}$$

$$= \frac{\bar{x} \cdot R_i(\bar{x} \alpha_1, ..., \bar{x} \alpha_n)}{k \cdot R(\bar{x} \alpha_1, ..., \bar{x} \alpha_n)}$$

$$= \frac{\bar{x} \cdot \bar{x}^{k-1} R_i(\alpha_1, ..., \alpha_n)}{k \cdot \bar{x}^k R(\alpha_1, ..., \alpha_n)}$$

$$= \frac{R_i(\alpha_1, ..., \alpha_n)}{kR(\alpha_1, ..., \alpha_n)},$$

where the second equality follows from $\alpha_i = \frac{x_i}{\bar{x}}$, and the third equality is the result of homogeneity. Notice that when $R$ is homogeneous, $B^R_i$ depends on portfolio weights only and not on the dollar amounts. The next proposition summarizes this result.

**Proposition 9** Assume $R(\cdot)$ is smooth and homogeneous of degree $k$, and suppose that $R(0, ..., 0) = 0$. Then, the systematic risk of asset $i = 1, ..., n$ depends on portfolio weights only, and is given by

$$B^R_i = \frac{R_i(\alpha_1, ..., \alpha_n)}{kR(\alpha_1, ..., \alpha_n)}. \tag{11}$$

### 3.3 Examples

All of our examples relate to homogeneous risk measures. Thus, in light of Proposition 9, we dispense with dollar amounts and state portfolio weights only. When the risk measure is the variance, $B^R_i$ boils down to the traditional CAPM beta, which is the slope-coefficient from a regression of individual returns on portfolio returns. To see this, assume $R(\cdot) = \text{Var}(\cdot)$ and consider portfolio weights $\alpha = (\alpha_1, ..., \alpha_n)$. Then,

$$R(\alpha_1, ..., \alpha_n) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \text{Cov} (\tilde{z}_i, \tilde{z}_j).$$
Hence,

\[ R_i(\alpha_1, ..., \alpha_n) = 2\text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{z}) , \]

and since the variance is homogeneous of degree \( k = 2 \), (11) becomes

\[ B_i^R \frac{\text{Var}(\alpha)}{R(\alpha_1, ..., \alpha_n)} = \frac{\text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{z})}{\text{Var}(\alpha)} . \]

We next illustrate how our general notion of systematic risk applies to other risk measures.

### 3.3.1 Central Moments

Let \( R(\cdot) = m_k(\cdot) \) for \( k \geq 2 \). That is, given any random return \( \tilde{z} \),

\[ R(\tilde{z}) = \mathbb{E}[(\tilde{z} - \mathbb{E}(\tilde{z}))^k] . \]

Consider portfolio weights \( \alpha = (\alpha_1, ..., \alpha_n) \). Then

\[ R(\alpha_1, ..., \alpha_n) = \mathbb{E}[(\alpha \cdot \tilde{z} - \alpha \cdot \mathbb{E}(\tilde{z}))^k] = \mathbb{E}\left[ \left( \sum_{i=1}^n \alpha_i(\tilde{z}_i - \mathbb{E}(\tilde{z}_i)) \right)^k \right] . \]

We then have,

**Proposition 10** Let \( R(\cdot) = m_k(\cdot) \) for \( k \geq 2 \). Then, the systematic risk is given by

\[ B_i^R = \frac{\text{Cov}(\tilde{z}_i, (\alpha \cdot (\tilde{z} - \mathbb{E}(\tilde{z})))^{k-1})}{\mathbb{E}[(\alpha \cdot (\tilde{z} - \mathbb{E}(\tilde{z})))^k]} . \]

That is, the systematic risk of asset \( i \) is proportional to the covariance of \( \tilde{z}_i \) with the \((k - 1)^{th}\) power of the demeaned portfolio return. In the special case of \( k = 2 \) (variance), this becomes

\[ B_i^R = \frac{\text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{z})}{\text{Var}(\alpha)} , \]

as expected.

Obviously, we can apply the same procedure to linear combinations of different moments. As an example, let

\[ R(\cdot) = \theta_1 m_2(\cdot) + \theta_2 (m_3(\cdot))^\frac{2}{3} \]

for some real \( \theta_1 \) and \( \theta_2 \). This risk measure captures both variance and skewness. Intuitively, it makes sense to have \( \theta_1 > 0 \) and \( \theta_2 < 0 \) to reflect aversion to both variance and negative skewness. Raising the skewness to the \( \frac{2}{3} \)-power ensures that the resulting measure is homogeneous. Then it is straightforward to show that

\[ B_i^R = \frac{\theta_1 \text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{z}) + \theta_2 (m_3(\alpha \cdot \tilde{z}))^{-\frac{1}{3}} \text{Cov}(\tilde{z}_i, (\alpha \cdot (\tilde{z} - \mathbb{E}(\tilde{z})))^2)}{\theta_1 m_2(\alpha \cdot \tilde{z}) + \theta_2 (m_3(\alpha \cdot \tilde{z}))^{\frac{2}{3}}} . \]
3.3.2 The AS and FH Measures

We now calculate the systematic risk of each asset based on the AS and FH measures. We have shown that both measures are smooth and homogeneous of degree 1. In addition, although \( R^{AS} (0, ..., 0) = R^{AS} (0) \) and \( R^{FH} (0, ..., 0) = R^{FH} (0) \) are not defined, they can be approximated using a limiting argument. Specifically, take any random return \( \tilde{z} \) satisfying \( E (\tilde{z}) > 0 \) and \( P (\{ \tilde{z} < 0 \}) > 0 \), and for both \( R (\cdot) = R^{AS} (\cdot) \) and \( R (\cdot) = R^{FH} (\cdot) \), we can then define \( R (0) \) by

\[
R (0) = \lim_{t \to 0} R (t \tilde{z}) = \lim_{t \to 0} t R (\tilde{z}) = R (\tilde{z}) \lim_{t \to 0} t = 0, 
\]

where the second equality follows since both the AS and the FH measures are homogeneous of degree 1. Hence, we can apply (11) to obtain the systematic risk of individual assets associated with the AS and FH measures. The results are stated in the following propositions.

Proposition 11 Let \( R (\cdot) = R^{AS} (\cdot) \). Then, the systematic risk is given by

\[
\mathcal{B}_i^R = \frac{E \left[ \exp \left( - \frac{\alpha \cdot \tilde{z}}{R (\alpha)} \right) \tilde{z}_i \right]}{E \left[ \exp \left( - \frac{\alpha \cdot \tilde{z}}{R (\alpha)} \right) \alpha \cdot \tilde{z} \right]}.
\]

Proposition 12 Let \( R (\cdot) = R^{FH} (\cdot) \). Then, the systematic risk is given by

\[
\mathcal{B}_i^R = \frac{E \left[ \frac{\tilde{z}_i}{R (\alpha)+\alpha \cdot \tilde{z}} \right]}{E \left[ \frac{\alpha \cdot \tilde{z}}{R (\alpha)+\alpha \cdot \tilde{z}} \right]}.
\]

4 An Equilibrium Setting

Traditionally, systematic risk is derived from an equilibrium setting known as the Capital Asset Pricing Model (CAPM). We will now present such a generalized setting. We will show that an equilibrium in this setting exists, and that any equilibrium exhibits two-fund separation. Furthermore, in equilibrium the standard CAPM formula holds where systematic risk coincides with the scaled Aumann-Shapley value introduced in Section 3.

4.1 Setup

Investors, Assets, and Timing. Assume a market with \( n + 1 \) assets \( \{0, ..., n\} \). Assets \( 1, ..., n \) are risky and pay a random amount denoted by \( (\tilde{y}_1, ..., \tilde{y}_n) \). Asset 0 is risk-free, paying an amount \( \tilde{y}_0 \) which is equal to some constant \( \tilde{y}_0 \neq 0 \) with probability 1. Denote \( \tilde{y} = (\tilde{y}_0, ..., \tilde{y}_n) \). There are \( \ell \) investors in the market, all of whom agree on the parameters of the model. The choice set of each investor is \( \mathbb{R}^{n+1} \),
where $\zeta^j \in \mathbb{R}^{n+1}$ represents the number of shares investor $j$ chooses in each asset $i = 0, ..., n$; i.e., $\zeta^j$ is a bundle of assets. Negative numbers represent short sales, and we impose no short-sale constraints. The initial endowment of investor $j$ is $e^j \in \mathbb{R}^{n+1}$.

We assume that $\sum_{i=1}^{\ell} \zeta^j_i > 0$ for $i = 1, ..., n$ and $\sum_{i=1}^{\ell} \zeta^j_0 = 0$. That is, risky assets are in positive net supply and the risk-free asset is in zero net supply. An allocation is an $\ell$-tuple $A = (\zeta^1, ..., \zeta^\ell)$ consisting of a bundle $\zeta^j \in \mathbb{R}^{n+1}$ for each investor. An allocation $A$ is attainable if $P_{\sum_{j=1}^{\ell} e^j} = P_{\sum_{j=1}^{\ell} \zeta^j}$, that is, if it clears the market. A price system is a vector $p = (p_0, ..., p_n)$ specifying a price for each asset. The random return of asset $i$ is then given by $\tilde{z}_i = \tilde{y}_i p_i$; where the return from the risk-free asset $\tilde{z}_0$ is equal to some constant $r_f$ with probability 1. Similar to the standard CAPM setting, there are two dates. At Date 0, investors trade with each other and prices are set. At Date 1, all random variables are realized.

**Risk and Preferences.** The traditional approach has investors having mean-variance preferences, i.e., they prefer higher mean and lower variance of investments. Instead, we will assume that investors have mean-risk preferences. Formally, fix a risk measure $R(\cdot)$. The utility that investor $j = 1, ..., \ell$ assigns to a bundle $\zeta^j \in \mathbb{R}^{n+1}$ is given by

$$U^j(\zeta) = V^j(E(\zeta \cdot \tilde{y}), R(\zeta \cdot \tilde{y})), \quad (12)$$

where $V^j$ is continuous, strictly increasing in its first argument (expected return) and strictly decreasing in its second argument (risk of return), and quasi-concave. An implication of quasi-concavity is that when plotted in mean-risk space, the upper contour of each indifference curve is convex. Similar to the standard mean-variance case, we will assume that a risk-free asset cannot be created synthetically from risky assets. That is, there is no redundant risky asset: for any $\zeta = (\zeta_0, \zeta_1, ..., \zeta_n) \in \mathbb{R}^{n+1}$ we have $R(\zeta \cdot \tilde{y}) \neq 0$ unless $(\zeta_1, ..., \zeta_n) = (0, ..., 0)$.

**Equilibrium.** An equilibrium is a pair $(p, A)$ where $p \neq 0$ is a price system and $A = (\zeta^1, ..., \zeta^\ell)$ is an attainable allocation, such that for each $j \in \{1, ..., \ell\}$, $p \cdot \zeta^j = p \cdot e^j$, and if $\zeta \in \mathbb{R}^{n+1}$ and $U^j(\zeta) > U^j(\zeta^j)$ then $p \cdot \zeta > p \cdot e^j$. In words, an equilibrium is a price system and an allocation that clears the market such that each investor optimizes subject to her budget constraint. The next lemma specifies conditions under which an equilibrium exists.

**Lemma 2** Suppose that $R(\cdot)$ is convex, smooth, and satisfies the risk-free property. Then, an equilibrium exists.

It is well known that the CAPM setting can yield negative or zero prices (see for example Nielsen (1992)). The reason for this is that preferences are not necessarily monotone. Specifically, the expected return to an investor’s bundle increases as she holds more shares of a (risky) asset, but so does the risk. It may well be that at some point, the additional expected return gained from adding more shares to the bundle is

\[\text{In the standard mean-variance case this condition corresponds to the variance-covariance matrix of risky assets being positive-definite.}\]
not sufficient to compensate for the increase in risk. If the equilibrium happens to fall in such a region then the asset becomes undesirable, rendering a negative price. For our following results we will need that prices are positive for all assets. The literature has suggested several ways to guarantee such an outcome. In Appendix II we provide one sufficient condition which follows Nielsen (1992). Other (and possibly weaker) sufficient conditions may be obtained, but are beyond the scope of this paper.

From now on we will only consider equilibria with positive prices. Given positivity of prices, naturally, each equilibrium induces a vector of random returns $\tilde{z}_i = \frac{y_i}{p_i}$, and so we can talk about the expected returns and the risk of the returns in equilibrium, as in the usual CAPM setting. We now study these returns.

### 4.2 A Generalized CAPM

#### 4.2.1 Geometry of Efficient Portfolios

Let $(p, A)$ be an equilibrium. The equilibrium allocation $(\zeta^1, ..., \zeta^\ell)$ naturally induces a portfolio for each investor $j$ given by $x^j = (x^j_0, ..., x^j_n)$ where $x^j_i = p_i \zeta^j_i$ is the amount invested in asset $i$, and where the vector of portfolio weights of investor $j$ is denoted by $\alpha^j$, and given by $\alpha^j_i = \frac{x^j_i}{\sum_{h=0}^n x^j_h}$. Let

$$\mu^j = \sum_{i=0}^n \alpha^j_i \mathbb{E}(\tilde{z}_i)$$

be the expected return obtained by investor $j$ in equilibrium. The next lemma shows that the standard procedure of “minimizing risk for a given expected return” applies to the equilibrium setting.

**Theorem 1** Suppose that $R(\cdot)$ is homogeneous. Then, in an equilibrium with positive prices, for all investors $j \in \{1, ..., \ell\}$, $\alpha^j$ is the unique solution to

$$\min_{\alpha \in \mathbb{R}^{n+1}} R(\alpha \cdot \tilde{z})$$

s.t.

$$\sum_{i=0}^n \alpha_i \mathbb{E}(\tilde{z}_i) = \mu^j.$$  

$$\sum_{i=0}^n \alpha_i = 1.$$  

Given this, we can now discuss the geometry of portfolios in the $\mu$-$R$ plane where the horizontal axis is the risk of the return of a portfolio ($R$) and the vertical axis is the expected return ($\mu$). The locus of portfolios minimizing risk for any given expected return is the boundary of the portfolio opportunity set. This set is convex in the $\mu$-$R$ plane whenever $R(\cdot)$ is a convex risk measure. This follows simply because the expectation operator is linear, implying that the line connecting any two portfolios in the $\mu$-$R$ plane lies to the right of the set of portfolios representing convex combinations
of these portfolios. Figure 1 illustrates two curves. The blue curve depicts the opportunity set of risky assets only. The red curve depicts portfolios minimizing risk for a given expected return, corresponding to Program (13). In general, both of these are defining convex sets, unlike in the special case of the standard deviation, where we have an extreme case of a straight line connecting the risk-free asset and risky portfolios. We say that a portfolio is efficient if it solves Program (13) for some $\mu^2$. Thus, the red curve in Figure 1 corresponds to the set of efficient portfolios.

4.2.2 Two-Fund Separation

We say that two-fund money separation holds if the equilibrium optimal portfolios for all investors can be spanned by the risk-free asset and a unique portfolio of risky assets only. The idea of two-fund separation was introduced by Tobin (1958). Since then the literature discussed different sufficient conditions for two-fund separation (see Cass and Stiglitz (1970), Ross (1978), and more recently Dybvig and Liu (2012)). These papers provide sufficient conditions for two-fund separation based on either restrictions on the return distributions or on the utility functions. Here we take a different approach, as we specify sufficient conditions for two-fund money separation in terms of properties of the risk measure.

Theorem 2 Consider an equilibrium with positive prices. Assume that $R(\cdot)$ is convex, homogeneous of some degree $k$, translation invariant, and satisfies the risk-free
property. Then, two-fund money separation holds. That is, there exists a unique portfolio with weights $\alpha^P$ such that $\alpha^P_0 = 0$, and for all investors $j \in \{1, \ldots, \ell\}$, the solution to Problem (13) is a linear combination of $\alpha^P$ and the risk-free asset.

The proof is very intuitive, and we show it here. Let $\alpha^1$ and $\alpha^2$ be solutions of Problem (13) for investors $j_1 \neq j_2$, respectively, and without loss of generality assume $j_1 = 1$ and $j_2 = 2$. Assume without loss of generality that both $\alpha^1$ and $\alpha^2$ have non-zero weights in some risky assets.\(^{10}\) By the risk-free property and by the non-redundancy assumption, $R(\alpha_j \cdot \tilde{z}) > 0$ for $j = 1, 2$. Hence, $\mu^j = E(\alpha_j \cdot \tilde{z}) > r_f$ for $j = 1, 2$, since otherwise $\alpha^j$ would be mean-risk dominated by the risk-free asset, and thus would not be optimal.

Now, consider all the linear combinations of these two portfolios with the risk-free asset. Since $R(\cdot)$ is assumed convex, the resulting curves are concave in the $\mu$-$R$ plane as illustrated in Figure 2. Note that both $\alpha^1$ and $\alpha^2$ can be presented as a linear combination of the risk-free asset and some portfolios $\alpha^{P_1}$ and $\alpha^{P_2}$ of risky assets only (i.e., $\alpha^P_0 = \alpha^{P_1}_0 = \alpha^{P_2}_0 = 0$). To show two-fund separation we need to show that $\alpha^{P_1} = \alpha^{P_2}$. Assume this is not the case. Then let $\hat{\alpha}^1$ be a portfolio of $\alpha^{P_2}$ and the risk-free asset such that $E(\hat{\alpha}^1 \cdot \tilde{z}) = \mu^1$. Similarly, let $\hat{\alpha}^2$ be a portfolio of $\alpha^{P_1}$ and the risk-free asset such that $E(\hat{\alpha}^2 \cdot \tilde{z}) = \mu^2$. By convexity of $R(\cdot)$, $\alpha^1$ and $\alpha^2$ are the unique solutions for Program (13) for $j = 1, 2$. Hence,

$$R(\hat{\alpha}^1 \cdot \tilde{z}) > R(\alpha^1 \cdot \tilde{z}) \text{ and } R(\hat{\alpha}^2 \cdot \tilde{z}) > R(\alpha^2 \cdot \tilde{z}).$$

Thus, as illustrated in Figure 2, the two curves must cross at least once. We will now show that such crossings are impossible.

By homogeneity, we have for any $\lambda > 0$,

$$R(\lambda \alpha^1 \cdot \tilde{z}) = \lambda^k R(\alpha^1 \cdot \tilde{z}) < \lambda^k R(\hat{\alpha}^1 \cdot \tilde{z}) = R(\lambda \hat{\alpha}^1 \cdot \tilde{z}),$$

which together with translation invariance implies

$$R(\lambda \alpha^1 \cdot \tilde{z} + (1 - \lambda) r_f) < R(\lambda \hat{\alpha}^1 \cdot \tilde{z} + (1 - \lambda) r_f).$$

This means that all linear combinations of $\alpha^1$ with the risk-free asset (with positive $\lambda$) lie strictly to the left of all linear combinations of $\hat{\alpha}^1$ with the risk-free asset. In particular, $\hat{\alpha}^2$ can be obtained as a linear combination of $\alpha^1$ with the risk-free asset by setting

$$\lambda = \frac{\mu^2 - r_f}{\mu^1 - r_f} > 0,$$

where the inequality follows since $\mu^j > r_f$ for $j = 1, 2$. But, using this $\lambda$ we obtain

$$R(\hat{\alpha}^2 \cdot \tilde{z}) < R(\alpha^2 \cdot \tilde{z}),$$

contradicting (14). Thus, two-fund separation must hold.

\(^{10}\)If only one investor holds non-zero weights in risky assets then two fund separation is trivial.
A corollary is that the unique portfolio $\alpha^P$ is efficient. Indeed, let $\mu^P = E(\alpha^P \cdot \tilde{z})$. Since in equilibrium all investors hold a linear combination of the risk-free asset and $\alpha^P$, and since $\mu^j = E(\alpha^j \cdot \tilde{z}) \geq r_f$ for all $j$ with strict inequality for some $j$, we have two cases: \(^{11}\) (i) all investors hold $\alpha^P$ with a non-negative weight, and $\mu^P > r_f$; or (ii) all investors hold $\alpha^P$ with a non-positive weight, and $\mu^P < r_f$. But, the second case is impossible since then the market cannot clear for at least one risky asset, which is held in positive weight in $\alpha^P$. Thus, $\mu^P > r_f$.

Now, assume that $\alpha' \neq \alpha^P$ solves Problem (13) for $\mu^j = \mu^P$. Then, $R(\alpha' \cdot \tilde{z}) < R(\alpha^P \cdot \tilde{z})$, and so by the same argument as in the proof of Theorem 2, all linear combinations of $\alpha'$ with the risk-free asset would have strictly lower risk than the corresponding linear combinations of $\alpha^P$ with the risk-free asset. This contradicts that $\alpha^P$ and the risk-free asset span all efficient portfolios. We thus have:

**Corollary 1** Under the conditions of Theorem 2, the portfolio $\alpha^P$ solves Problem (13) for some $\mu^P > r_f$.

Let $x^M_i = \sum_{j=1}^{\ell} x^j_i$ be the total amount invested in asset $i$. We call $x^M = (x^M_0, ..., x^M_n)$ the market portfolio. Note that since the risk-free asset is in zero net supply we have $x^M_0 = 0$. Let $\alpha^M$ be the corresponding portfolio weights, where in

\(^{11}\) If all investors choose the risk-free asset then the market for risky assets cannot clear.
particular $\alpha^M_0 = 0$. By Theorem 2, in equilibrium, the market portfolio is equal to $\alpha^P$ the unique portfolio of risky assets that together with the risk-free asset spans all efficient portfolios. Moreover, by corollary 1, the market portfolio is efficient, and its expected return is strictly higher than $r_f$.

**Corollary 2** Under the conditions of Theorem 2, the market portfolio solves Problem (13) for some $\mu^M > r_f$.

### 4.2.3 A Generalized Security Market Line

In the traditional CAPM framework, the security market line describes the equilibrium relation between the expected returns of individual assets and the market expected return. In particular, the expected return of any asset in excess of the risk-free rate is proportional to the excess market expected return, with the proportion being equal to the traditional beta. The following theorem provides sufficient conditions under which a similar relation holds with respect to our Aumann-Shapley based systematic risk measure.

**Theorem 3** Consider an equilibrium with positive prices and let $\alpha^M$ be the market portfolio. Assume that $R(\cdot)$ is convex, homogeneous of some degree $k$, translation invariant, smooth and satisfies the risk-free property. Then, for each asset $i = 1, \ldots, n$,

$$E(\tilde{z}_i) = r_f + B_i^R (E(\alpha^M \cdot \tilde{z}) - r_f),$$

where

$$B_i^R = \frac{R_i(\alpha^M)}{kR(\alpha^M)},$$

is the Aumann-Shapley based measure of systematic risk.

As an illustration, consider

$$R(\cdot) = \theta m_2(\cdot) + (1 - \theta) \sqrt{m_4(\cdot)}$$

for $\theta \in [0, 1]$. As discussed in Section 2.2.1, this risk measure captures both variance and tail risk, and it satisfies all of the conditions in Theorem 3. Then it is straightforward to verify that

$$B_i^R = \frac{\theta \text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{z}) + (1 - \theta)(m_4(\alpha \cdot \tilde{z}))^{-\frac{1}{2}} \text{Cov} \left( \tilde{z}_i, (\alpha \cdot (\tilde{z} - E(\tilde{z})))^3 \right)}{\theta m_2(\alpha \cdot \tilde{z}) + (1 - \theta)(m_4(\alpha \cdot \tilde{z}))^{\frac{1}{2}}}. $$

### 4.2.4 Discussion

Before concluding this section we point out two additional issues. First, while we described sufficient conditions under which two-fund money separation holds and use this to argue that the market portfolio is efficient, these conditions are by no means necessary. Weaker conditions that guarantee two-fund money separation may exist.
Further, even when two-fund money separation fails, it does not necessarily mean that market efficiency is rejected. The literature explores market efficiency from both theoretical (see, for example, Dybvig and Ross (1982)) and empirical (see, for example, Levy and Roll (2010)) views. Our generalized SML remains valid as long as we have evidence that the market portfolio is mean-risk efficient.

Second, the SML pricing formula (15) can indeed be applied not only to the market portfolio, but with respect to any efficient portfolio with a slightly more complex form. Specifically, suppose that $\alpha^*$ represents a portfolio that is mean-risk efficient. Then, it can be easily seen from the proof of Theorem (3) that in equilibrium, we have for each asset $i$,

$$E(\tilde{z}_i) = r_f + \frac{R_i(\alpha^*)}{kR(\alpha^*) - \alpha_0^* R_0(\alpha^*)} (E(\alpha^* \cdot \tilde{z}) - r_f).$$

In particular, if $\alpha^*$ involves a zero weight in the risk-free asset ($\alpha_0^* = 0$), as in the case of the market portfolio, or, if adding a constant return does not change the risk of $\alpha^*$ locally ($R_0(\alpha^*) = 0$), then this formula can be reduced to (15).

5 Conclusion

In this paper we define a broad notion of risk measures, which encapsulate some or all uncertainty of a random variable using one real number. We develop a general framework based on the Aumann-Shapley solution concept to measure the systematic risk of an asset, capturing the marginal contribution of this asset to the risk of a portfolio. We then derive sufficient conditions for two-fund money separation in an equilibrium setting of an asset market with investors having mean-risk preferences. Furthermore, we establish a generalization of the CAPM in which our measure of systematic risk emerges naturally as the generalized beta in the security market line.

Our Aumann-Shapley based systematic risk applies to a wide variety of risk measures, requiring of them only smoothness and zero risk for zero investment. Although our generalized CAPM imposes additional conditions, including convexity, homogeneity, translation invariance, and the risk-free property, we are still left with an extensive class of risk measures. Indeed, this class is sufficiently broad to potentially account for high distribution moments, rare disasters, downside risk, as well as many other aspects of risk. Future research may direct at developing weaker conditions that further expand the set of applicable risk measures.

Finally, which risk measures better capture the risk attributes of the investors is ultimately an empirical question. Our framework therefore provides the foundation for testing the appropriateness of risk measures and consequently selecting those that correspond well with the data.

References


**Appendix I**

**Proof of Proposition 1:** The homogeneity, translation invariance, smoothness and risk-free properties are obvious. The convexity of $m_k(\bar{z})$ follows from the more general result that the normalized $k^{th}$ moment $w_k(\bar{z})$ is convex, which will be proved in Proposition 2. ■
Proof of Proposition 2: The homogeneity, translation invariance, smoothness and risk-free properties are obvious. We now prove the convexity of \( w_k (z) \).

What we need to show is that for any random returns \( \tilde{z}_1 \) and \( \tilde{z}_2 \), and any \( 0 \leq \lambda \leq 1 \),
\[
w_k (\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2) \leq \lambda w_k (\tilde{z}_1) + (1 - \lambda) w_k (\tilde{z}_2).
\]
(18)

Letting \( \tilde{z}_1 = \tilde{z}_1 - E(\tilde{z}_1) \) and \( \tilde{z}_2 = \tilde{z}_2 - E(\tilde{z}_2) \), (18) can be rewritten as
\[
\left( E \left[ (\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2)^k \right] \right)^{\frac{1}{k}} \leq \lambda \left( E \left[ z_1^k \right] \right)^{\frac{1}{k}} + (1 - \lambda) \left( E \left[ z_2^k \right] \right)^{\frac{1}{k}}.
\]
(19)

From the binomial formula we know that for any two numbers \( p \) and \( q \),
\[
(p + q)^k = \sum_{i=0}^{k} \binom{k}{i} p^{k-i} q^i.
\]

Applying this to the LHS of (19) implies that we need to show
\[
\left( \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} (1 - \lambda)^i \left( \frac{z_1^{k-i} z_2^i}{z_1^{k-i} z_2^i} \right) \right)^{\frac{1}{k}} \leq \lambda \left( E \left[ |z_1|^k \right] \right)^{\frac{1}{k}} + (1 - \lambda) \left( E \left[ |z_2|^k \right] \right)^{\frac{1}{k}}.
\]

Since \( k \) is even, replacing each \( \tilde{z}_1 \) and \( \tilde{z}_2 \) with \( |\tilde{z}_1| \) and \( |\tilde{z}_2| \) will not affect the RHS, but it might increase the LHS. So, it is sufficient to show that
\[
\left( \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} (1 - \lambda)^i \left( \frac{|z_1|^{k-i} |z_2|^i}{|z_1|^{k-i} |z_2|^i} \right) \right)^{\frac{1}{k}} \leq \lambda \left( E \left[ |z_1|^k \right] \right)^{\frac{1}{k}} + (1 - \lambda) \left( E \left[ |z_2|^k \right] \right)^{\frac{1}{k}}.
\]

Since both sides are positive we can raise both sides to the \( k \)th power, maintaining the inequality. Thus, it would be sufficient to show
\[
\sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} (1 - \lambda)^i \left( \frac{|z_1|^{k-i} |z_2|^i}{|z_1|^{k-i} |z_2|^i} \right) \leq \lambda \left( E \left[ |z_1|^k \right] \right)^{\frac{1}{k}} + (1 - \lambda) \left( E \left[ |z_2|^k \right] \right)^{\frac{1}{k}}.
\]

Applying the binomial formula to the RHS implies that it would be sufficient to show
\[
\sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} (1 - \lambda)^i \left( \frac{|z_1|^{k-i} |z_2|^i}{|z_1|^{k-i} |z_2|^i} \right) \leq \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} (1 - \lambda)^i \left( E \left[ |z_1|^k \right] \right)^{\frac{k-i}{k}} \left( E \left[ |z_2|^k \right] \right)^{\frac{i}{k}}.
\]

To establish this inequality we will show that it actually holds term by term. That is, it is sufficient to show that for each \( i = 0, \ldots, k \),
\[
E \left( \frac{|z_1^{k-i} z_2^i|}{|z_1|^{k-i} |z_2|^i} \right) \leq \left( E \left[ |z_1|^k \right] \right)^{\frac{k-i}{k}} \left( E \left[ |z_2|^k \right] \right)^{\frac{i}{k}}.
\]
To see this, note that it is equivalent to show that
\[
E \left( \frac{|z_1^{k-i} z_2^i|}{|z_1|^{k-i} |z_2|^i} \right) \leq \left( E \left[ |z_1|^{k-1} \right] \right)^{\frac{k-i}{k}} \left( E \left[ |z_2|^k \right] \right)^{\frac{i}{k}}.
\]
Now, denote \( Z_1 = \hat{z}_1^k \), \( Z_2 = \hat{z}_2^i \), \( p = \frac{k}{k-i} \) and \( q = \frac{k}{k} \). Note that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, what we need to show is
\[
E (|Z_1 Z_2|) \leq E (|Z_1|^p)^{\frac{1}{p}} E (|Z_2|^q)^{\frac{1}{q}}.
\]
But, this is immediate from Holder’s inequality, and we are done. 

**Proof of Proposition 5:** Given any random return \( \hat{z} \) and for all \( \lambda > 0 \), since \( \lambda \hat{z} \) scales the distribution of \( \hat{z} \) by a factor of \( \lambda \), clearly we have
\[
\text{VaR}_\alpha (\lambda \hat{z}) = \lambda \text{VaR}_\alpha (\hat{z}).
\]
Hence, \( \text{VaR}_\alpha (\hat{z}) \) is homogeneous of degree 1.

Given any random return \( \hat{z} \) and for any constant \( \lambda \in \mathbb{R} \), since \( \hat{z} + \lambda \) shifts the distribution of \( \hat{z} \) by a distance of \( \lambda \), it then follows that
\[
\text{VaR}_\alpha (\hat{z} + \lambda) = \text{VaR}_\alpha (\hat{z}) - \lambda.
\]
This implies that \( \text{VaR}_\alpha (\hat{z}) \) is translation invariant.

To show smoothness, suppose that \( \hat{z} \) follows a joint distribution with a continuously differentiable probability density function \( f(z_1, \ldots, z_n) \). Then, using definition and applying the convolution formula, we have
\[
\text{VaR}_\alpha (x) = \frac{1}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1, \ldots, z_{n-1}, r - \sum_{i=1}^{n-1} x_i z_i) dz_1 \ldots dz_{n-1} dr} = \alpha,
\]
which implicitly defines \( \text{VaR}_\alpha (x) \) as a function of \( x \). Let
\[
h (\text{VaR}_\alpha (x), x) = \int_{-\infty}^{-\text{VaR}_\alpha (x)} \cdots \int_{-\infty}^{+\infty} f(z_1, \ldots, z_{n-1}, r - \sum_{i=1}^{n-1} x_i z_i) dz_1 \ldots dz_{n-1} dr - \alpha.
\]
Since
\[
\frac{\partial h (\text{VaR}_\alpha (x), x)}{\partial \text{VaR}_\alpha (x)} = - \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(z_1, \ldots, z_{n-1}, \frac{\text{VaR}_\alpha (x) - \sum_{i=1}^{n-1} x_i z_i}{x_n}) dz_1 \ldots dz_{n-1} < 0,
\]
we know by the implicit function theorem that \( \frac{\partial \text{VaR}_\alpha (x)}{\partial x_i} \) uniquely exists and is continuous for all \( i = 1, \ldots, n \). 

**Proof of Proposition 6:** The proof of homogeneity and translation invariance of \( \text{ExVaR}_\alpha (\hat{z}) \) is parallel to the proof of the same properties for \( \text{VaR}_\alpha (\hat{z}) \).

To show convexity, we first prove that \( \text{ExVaR}_\alpha (\hat{z}) \) is subadditive. That is, for any two random returns \( \hat{z}_1 \) and \( \hat{z}_2 \),
\[
\text{ExVaR}_\alpha (\hat{z}_1 + \hat{z}_2) \leq \text{ExVaR}_\alpha (\hat{z}_1) + \text{ExVaR}_\alpha (\hat{z}_2).
\]
(20)
By definition, for any random return \( \hat{z} \), \( \text{ExVaR}_\alpha (\hat{z}) \) can be expressed as
\[
\text{ExVaR}_\alpha (\hat{z}) = -\frac{1}{\alpha} \int_{\{\omega: \hat{z} \leq -\text{VaR}_\alpha (\hat{z})\}} \hat{z} dP (\omega).
\]

Define \( \tilde{z}_3 = \tilde{z}_1 + \tilde{z}_2 \). Let \( \Omega_i = \{ \omega \in \Omega : \tilde{z}_i \leq -\text{VaR}_\alpha (\tilde{z}_i) \} \) for \( i = 1, 2, 3 \). Then, (20) is equivalent to
\[
\int_{\Omega_3} \tilde{z}_3 dP (\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP (\omega) + \int_{\Omega_2} \tilde{z}_2 dP (\omega),
\]
which can be rewritten as
\[
\int_{\Omega_3} \tilde{z}_1 dP (\omega) + \int_{\Omega_3} \tilde{z}_2 dP (\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP (\omega) + \int_{\Omega_2} \tilde{z}_2 dP (\omega).
\]
This is true if we have
\[
\int_{\Omega_3} \tilde{z}_1 dP (\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP (\omega), \tag{21}
\]
and
\[
\int_{\Omega_3} \tilde{z}_2 dP (\omega) \geq \int_{\Omega_2} \tilde{z}_2 dP (\omega). \tag{22}
\]
For brevity, we will only prove (21) below. The proof of (22) is parallel.

To prove (21), define \( \Omega_4 = \{ \omega \in \Omega : \tilde{z}_1 \leq -\text{VaR}_\alpha (\tilde{z}_1), \tilde{z}_3 \leq -\text{VaR}_\alpha (\tilde{z}_3) \} \), \( \Omega_5 = \{ \omega \in \Omega : \tilde{z}_1 \leq -\text{VaR}_\alpha (\tilde{z}_1), \tilde{z}_3 > -\text{VaR}_\alpha (\tilde{z}_3) \} \) and \( \Omega_6 = \{ \omega \in \Omega : \tilde{z}_1 > -\text{VaR}_\alpha (\tilde{z}_1), \tilde{z}_3 \leq -\text{VaR}_\alpha (\tilde{z}_3) \} \). Since \( \Omega_4 \cap \Omega_5 = \emptyset \), \( \Omega_4 \cup \Omega_5 = \Omega_1 \), and \( \Omega_4 \cap \Omega_6 = \emptyset \), \( \Omega_4 \cup \Omega_6 = \Omega_3 \), we have that
\[
\int_{\Omega_1} dP (\omega) = \int_{\Omega_4} dP (\omega) + \int_{\Omega_5} dP (\omega),
\]
and
\[
\int_{\Omega_3} dP (\omega) = \int_{\Omega_4} dP (\omega) + \int_{\Omega_6} dP (\omega).
\]
By the definition of VaR, we know
\[
\int_{\Omega_1} dP (\omega) = \int_{\Omega_3} dP (\omega) = \alpha.
\]
Thus, we obtain
\[
\int_{\Omega_5} dP (\omega) = \int_{\Omega_6} dP (\omega). \tag{23}
\]
Similarly, we know
\[
\int_{\Omega_1} \tilde{z}_1 dP (\omega) = \int_{\Omega_4} \tilde{z}_1 dP (\omega) + \int_{\Omega_5} \tilde{z}_1 dP (\omega),
\]
and
\[
\int_{\Omega_3} \tilde{z}_1 dP (\omega) = \int_{\Omega_4} \tilde{z}_1 dP (\omega) + \int_{\Omega_6} \tilde{z}_1 dP (\omega).
\]
Hence,

\[
\int_{\Omega_5} z_1 dP(\omega) - \int_{\Omega_6} z_1 dP(\omega) = \int_{\Omega_5} [\operatorname{VaR}_\alpha(z_1)] dP(\omega) - \int_{\Omega_6} [\operatorname{VaR}_\alpha(z_1)] dP(\omega)
\]

\[
\leq - \operatorname{VaR}_\alpha(z_1) \left[ \int_{\Omega_5} dP(\omega) - \int_{\Omega_6} dP(\omega) \right],
\]

where the inequality follows from \( z_1 \leq - \operatorname{VaR}_\alpha(z_1) \) when \( \omega \in \Omega_5 \) and \( z_1 > - \operatorname{VaR}_\alpha(z_1) \) when \( \omega \in \Omega_6 \). By (23), we have

\[
\int_{\Omega_5} dP(\omega) - \int_{\Omega_6} dP(\omega) = 0,
\]

which implies

\[
\int_{\Omega_5} z_1 dP(\omega) - \int_{\Omega_6} z_1 dP(\omega) \leq 0.
\]

Therefore, (21) is obtained, and hence \( \operatorname{ExVaR}_\alpha(\bar{z}) \) is subadditive.

Then, convexity follows immediately from homogeneity and subadditivity, since for any \( \lambda \in [0, 1] \),

\[
\operatorname{ExVaR}_\alpha(\lambda z_1 + (1 - \lambda) z_2) \leq \operatorname{ExVaR}_\alpha(\lambda z_1) + \operatorname{ExVaR}_\alpha((1 - \lambda) z_2)
\]

\[
= \lambda \operatorname{ExVaR}_\alpha(z_1) + (1 - \lambda) \operatorname{ExVaR}_\alpha(z_2).
\]

Finally, let us prove smoothness. Suppose that \( \bar{z} \) follows a joint distribution with a continuously differentiable probability density function \( f(z_1, ..., z_n) \). Then, using definition and applying the convolution formula, we have

\[
\operatorname{ExVaR}_\alpha(x) = -\frac{1}{\alpha} \int_{-\infty}^{-\operatorname{VaR}_\alpha(x)} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} r f(z_1, ..., z_{n-1}, r - \sum_{i=1}^{n-1} x_i z_i) dz_1 ... dz_{n-1} dr.
\]

Without loss of generality, we will focus on \( \frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_1} \), and \( \frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_i} \) for \( i = 2, ..., n \) can be derived in a similar manner.

Differentiating \( \operatorname{ExVaR}_\alpha(x) \) with respect to \( x_1 \) using the Leibniz integral rule yields

\[
\frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_1} = -\frac{1}{\alpha} \left[ \int_{-\infty}^{-\operatorname{VaR}_\alpha(x)} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} r f_n(z_1, ..., z_{n-1}, r - \sum_{i=1}^{n-1} x_i z_i) \left( -\frac{z_1}{x_n} \right) dz_1 ... dz_{n-1} dr + \operatorname{VaR}_\alpha(x) \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} f(z_1, ..., z_{n-1}, -\operatorname{VaR}_\alpha(x) - \sum_{i=1}^{n-1} x_i z_i) dz_1 ... dz_{n-1} \cdot \frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_1} \right],
\]

where \( f_n \) is the derivative of \( f \) with respect to the \( n^{th} \) argument. We already know from Proposition 5 that \( \frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_1} \) uniquely exists and is continuous. Therefore, \( \frac{\partial \operatorname{ExVaR}_\alpha(x)}{\partial x_1} \) is also continuous, and thus \( \operatorname{ExVaR}_\alpha(x) \) is continuously differentiable with respect to \( x \).
Proof of Proposition 7: It is proved in Aumann and Serrano (2008) and Foster and Hart (2009) that both measures are subadditive and homogeneous of degree 1, which together imply convexity. Hence, the only thing that needs to be proved is smoothness.

Let us work with $R^{AS}$ first. For any portfolio $x$, whose return is given by $x \cdot \tilde{z}$, the $R^{AS}$ measure is given by

$$E \left[ \exp \left( -\frac{x \cdot \tilde{z}}{R^{AS}(x)} \right) \right] = 1.$$  

Define

$$h^{AS}(R^{AS}, x) = E \left[ \exp \left( -\frac{x \cdot \tilde{z}}{R^{AS}(x)} \right) \right] - 1.$$  

It is shown in Kadan and Liu (2012) that $\frac{\partial h^{AS}}{\partial P^{AS}} > 0$ where $P^{AS} = \frac{1}{R^{AS}}$. Therefore,

$$\frac{\partial h^{AS}}{\partial R^{AS}} = \frac{\partial h^{AS}}{\partial P^{AS}} \cdot \frac{\partial P^{AS}}{\partial R^{AS}} = -\frac{1}{(R^{AS})^2} \frac{\partial h^{AS}}{\partial P^{AS}} < 0.$$  

This allows us to apply the implicit function theorem, according to which $\frac{\partial P^{AS}(x)}{\partial x_i}$ uniquely exists and is continuous for any $i = 1, \ldots, n$.

The proof is similar for $R^{FH}$. Given any portfolio $x$, the $R^{FH}$ measure is given by

$$E \left[ \log \left( 1 + \frac{x \cdot \tilde{z}}{R^{FH}(x)} \right) \right] = 0.$$  

Define

$$h^{FH}(R^{FH}, x) = E \left[ \log \left( 1 + \frac{x \cdot \tilde{z}}{R^{FH}(x)} \right) \right].$$  

It is shown in Kadan and Liu (2012) that $\frac{\partial h^{FH}}{\partial P^{FH}} < 0$ where $P^{FH} = \frac{1}{R^{FH}}$. Therefore,

$$\frac{\partial h^{FH}}{\partial R^{FH}} = \frac{\partial h^{FH}}{\partial P^{FH}} \cdot \frac{\partial P^{FH}}{\partial R^{FH}} = -\frac{1}{(R^{FH})^2} \frac{\partial h^{FH}}{\partial P^{FH}} > 0.$$  

This allows us to apply the implicit function theorem, according to which $\frac{\partial P^{FH}(x)}{\partial x_i}$ uniquely exists and is continuous for any $i = 1, \ldots, n$.

Proof of Proposition 10: The partial derivatives of $R(\cdot)$ with respect to the portfolio weights are given by $\forall i$,

$$R_i(\alpha_1, \ldots, \alpha_n) = kE \left[ (\tilde{z}_i - E(\tilde{z}_i)) \left( \sum_{j=1}^{n} \alpha_j (\tilde{z}_j - E(\tilde{z}_j)) \right)^{k-1} \right]$$  

$$= kE \left[ (\tilde{z}_i - E(\tilde{z}_i)) (\alpha \cdot \tilde{z} - \alpha \cdot E(\tilde{z}))^{k-1} \right]$$  

$$= kCov \left( \tilde{z}_i, (\alpha \cdot \tilde{z} - \alpha \cdot E(\tilde{z}))^{k-1} \right).$$
Thus, the systematic risk of asset \( i \) becomes

\[
B_i^R = \frac{\text{Cov}(\tilde{z}_i, (\alpha \cdot \tilde{z} - \alpha \cdot E(\tilde{z}))^{k-1})}{E[(\alpha \cdot \tilde{z} - \alpha \cdot E(\tilde{z}))^k]}.
\]

\[ \square \]

**Proof of Proposition 11:** When \( R(\cdot) = R^{AS}(\cdot) \), \( R(\alpha) \) is implicitly determined by

\[
E\left[ \exp\left( -\frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \right] = 1.
\]

By the implicit function theorem,

\[
R_i(\alpha) = \frac{E\left[ \exp\left( -\frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \frac{\tilde{z}_i}{R(\alpha)} \right]}{E\left[ \exp\left( -\frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \alpha \cdot \tilde{z} \right]}.
\]

Since the AS measure is homogeneous of degree 1, (11) becomes

\[
B_i^R = \frac{E\left[ \exp\left( -\frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \tilde{z}_i \right]}{E\left[ \exp\left( -\frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \alpha \cdot \tilde{z} \right]}.
\]

\[ \square \]

**Proof of Proposition 12:** When \( R(\cdot) = R^{FH}(\cdot) \), \( R(\alpha) \) is implicitly determined by

\[
E\left[ \log\left( 1 + \frac{\alpha \cdot \tilde{z}}{R(\alpha)} \right) \right] = 0.
\]

The implicit function theorem yields

\[
R_i(\alpha) = \frac{E\left[ \frac{1}{1 + \frac{\alpha \cdot \tilde{z}}{R(\alpha)}} \tilde{z}_i \right]}{E\left[ \frac{1}{1 + \frac{\alpha \cdot \tilde{z}}{R(\alpha)}} \frac{\alpha \cdot \tilde{z}}{R(\alpha)^2} \right]} = \frac{R(\alpha) \cdot E\left[ \frac{\tilde{z}_i}{R(\alpha) + \alpha \cdot \tilde{z}} \right]}{E\left[ \frac{\alpha \cdot \tilde{z}}{R(\alpha) + \alpha \cdot \tilde{z}} \right]}.
\]

The systematic risk of asset \( i \) for the FH measure is hence given by

\[
B_i^R = \frac{E\left[ \frac{\tilde{z}_i}{R(\alpha) + \alpha \cdot \tilde{z}} \right]}{E\left[ \frac{\alpha \cdot \tilde{z}}{R(\alpha) + \alpha \cdot \tilde{z}} \right]}.
\]
Proof of Lemma 2: Our setting is a special case of the setting in Nielsen (1989). To show the existence of equilibrium Nielsen requires that preferences satisfy the following three conditions: (i) each investor’s choice set is closed and convex, and contains her initial endowment; (ii) The set of \( \{ \zeta \in \mathbb{R}^{n+1} : U^j (\zeta) \geq U^j (\zeta') \} \) is closed for all \( \zeta' \in \mathbb{R}^{n+1} \) and for all \( j = \{1, \ldots, \ell\} \); (iii) If \( \zeta, \zeta' \in \mathbb{R}^{n+1} \) and \( U^j (\zeta') > U^j (\zeta) \), then \( U^j (t\zeta' + (1-t)\zeta) > U^j (\zeta) \) for all \( t \) in \( (0, 1) \).

Condition (i) is satisfied in our setting since the choice set of each investor is \( \mathbb{R}^{n+1} \), which is closed and convex, and contains \( e^j \) for all \( j \). Condition (ii) holds since \( V \) is assumed continuous and \( R \) is assumed smooth, and so their composition is continuous. Condition (iii) follows since \( V(\cdot) \) is quasi-concave, strictly increasing in its first argument and strictly decreasing in its second argument, and \( R(\cdot) \) is a convex risk measure.

Given these properties of the preferences, Nielsen (1989) establishes two conditions as sufficient for the existence of a quasi-equilibrium: (i) positive semi-independence of directions of improvement, and (ii) non-satiation at Pareto attainable portfolios. Condition (i) follows in our setting as in Nielsen (1990, Proposition 1) since in our setting all investors agree on all parameters of the problem (in particular on the expected returns), and due to the non-redundancy of risky assets assumption. To see why condition (ii) holds in our setting note that we assume the existence of a risk-free asset paying a non-zero payoff with probability 1. Since \( R(\cdot) \) satisfies the risk-free property, we have that \( R(\tilde{z}_1 + \tilde{z}_2) \leq R(\tilde{z}_1) \) whenever \( \tilde{z}_2 \) is \( R \)-risk-free with \( P(\{ \tilde{z}_2 > 0 \}) = 1 \). Thus, adding a positive risk-free asset can only (weakly) reduce risk. It follows that we can always add this positive risk-free asset to any bundle \( \zeta \), strictly increasing the expected return while weakly decreasing risk. This implies that in our model there is no satiation globally. Thus, a quasi-equilibrium exists in our setting. Moreover, any quasi-equilibrium is, in fact, an equilibrium in our setting. This follows from the conditions in Nielsen (1989 p. 469). Indeed, in our setting each investor’s choice set is convex and unbounded, and the set \( \{ \zeta \in \mathbb{R}^{n+1} : U^j (\zeta) > U^j (\zeta') \} \) is open for all \( j \) and \( \zeta' \in \mathbb{R}^{n+1} \).

Proof of Theorem 1: Suppose that the equilibrium bundle of investor \( j \) is \( \zeta^j \). Let
\[ x^j = \sum_{i=0}^{n} x^j_i = p \cdot \zeta^j \] be the total dollar amount of investment of investor \( j \). Then,

\[
U^j (\zeta^j) = V^j \left( E \left( \sum_{i=0}^{n} \frac{\zeta^j_i y_i}{\tilde{x}^j} \right), R \left( \sum_{i=0}^{n} \frac{\zeta^j_i y_i}{\tilde{x}^j} \right) \right)
\]

(24)

\[
= V^j \left( \tilde{x}^j E \left( \sum_{i=0}^{n} \frac{\zeta^j_i \tilde{y}_i}{\tilde{x}^j \tilde{p}_i} \right), (\tilde{x}^j)^k R \left( \sum_{i=0}^{n} \frac{\zeta^j_i \tilde{y}_i}{\tilde{x}^j \tilde{p}_i} \right) \right)
\]

\[
= V^j \left( \tilde{x}^j E \left( \sum_{i=0}^{n} \frac{x^j_i z_i}{x^j} \right), (\tilde{x}^j)^k R \left( \sum_{i=0}^{n} \frac{x^j_i z_i}{x^j} \right) \right)
\]

\[
= V^j \left( \tilde{x}^j E \left( \sum_{i=0}^{n} \alpha_i^j z_i \right), (\tilde{x}^j)^k R \left( \sum_{i=0}^{n} \alpha_i^j z_i \right) \right)
\]

\[
= V^j \left( \tilde{x}^j E (\alpha^j \tilde{z}), (\tilde{x}^j)^k R (\alpha^j \tilde{z}) \right),
\]

where \( k \) is the degree of homogeneity of \( R(\cdot) \). From the definition of equilibrium, each investor chooses \( \zeta^j \) to maximize \( U^j (\zeta^j) \) subject to \( \tilde{x}^j \leq p \cdot e^j \), where in equilibrium \( \tilde{x}^j = p \cdot e^j \) is given. From (24) and since \( V^j \) is strictly increasing in the first argument and strictly decreasing in the second argument, we have that for any positive \( \tilde{x}^j \), \( U^j (\zeta^j) \) is strictly increasing in \( E (\alpha^j \tilde{z}) \) and strictly decreasing in \( R (\alpha^j \tilde{z}) \). Therefore, in equilibrium, \( \alpha^j \) must minimize \( R (\alpha \cdot \tilde{z}) \) for a given level of expected return \( E (\alpha^j \tilde{z}) \), and thus solve Problem (13). The solution is unique since we assumed that \( R(\cdot) \) is a convex risk measure, and so \( R (\alpha \cdot \tilde{z}) \) is convex as a function of \( \alpha \). \( \square \)

**Proof of Theorem 3:** Since \( R(\cdot) \) is smooth and by Lemma 2 the solution to Problem (13) for some \( \mu^j = \mu \) is determined by the first order conditions. To solve this program, form the Lagrangian

\[
\mathcal{L} (\alpha) = R (\alpha) - \lambda \left( \sum_{i=1}^{n} \alpha_i E (\tilde{z}_i) + \left( 1 - \sum_{i=1}^{n} \alpha_i \right) r_f - \mu \right),
\]

where \( \lambda \) is a Lagrangian multiplier. The first order condition states that \( \forall i = 1, \ldots, n, \)

\[
R_i (\alpha^*) - \lambda (E (\tilde{z}_i) - r_f) = 0,
\]

(25)

where \( \alpha^* \) represents the optimal portfolio. Multiplying by \( \alpha^*_i \), we obtain

\[
\alpha^*_i R_i (\alpha^*) - \lambda \alpha^*_i (E (\tilde{z}_i) - r_f) = 0.
\]

Summing up over \( i = 1, \ldots, n \) yields

\[
\sum_{i=1}^{n} \alpha^*_i R_i (\alpha^*) - \lambda \sum_{i=1}^{n} \alpha^*_i (E (\tilde{z}_i) - r_f) = 0.
\]
Adding and subtracting $\alpha_0^* R_0 (\alpha^*)$ gives
\[
\sum_{i=0}^{n} \alpha_i^* R_i (\alpha^*) - \alpha_0^* R_0 (\alpha^*) - \lambda \sum_{i=1}^{n} \alpha_i^* (E (\tilde{z}_i) - r_f) = 0.
\]

By Euler’s homogeneous function theorem, we can rewrite this as
\[
kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*) - \lambda \sum_{i=1}^{n} \alpha_i^* (E (\tilde{z}_i) - r_f) = 0.
\]

Adding and subtracting $r_f$ inside the summation yields
\[
kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*) - \lambda \left( \sum_{i=1}^{n} \alpha_i^* E (\tilde{z}_i) + r_f \left( 1 - \sum_{i=1}^{n} \alpha_i^* \right) - r_f \right) = 0,
\]
which by the optimization constraint is equivalent to
\[
kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*) - \lambda (\mu - r_f) = 0.
\]

Hence,
\[
\lambda = \frac{kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*)}{\mu - r_f}.
\]

Plugging (26) into (25) gives
\[
R_i (\alpha^*) - \frac{kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*)}{\mu - r_f} (E (\tilde{z}_i) - r_f) = 0,
\]
which can be rewritten as
\[
E (\tilde{z}_i) = r_f + \frac{R_i (\alpha^*)}{kR (\alpha^*) - \alpha_0^* R_0 (\alpha^*)} (\mu - r_f).
\]

In particular, choose $\mu = \mu^M$, the expected return of the market portfolio. By Corollary 2, the market portfolio is efficient, and hence
\[
E (\tilde{z}_i) = r_f + \frac{R_i (\alpha^M)}{kR (\alpha^M) - \alpha_0^M R_0 (\alpha^M)} (\mu^M - r_f).
\]

Moreover, since $\alpha_0^M = 0$ (the risk free asset is in zero net supply) we obtain
\[
E (\tilde{z}_i) = r_f + \frac{R_i (\alpha^M)}{kR (\alpha^M)} (\mu^M - r_f),
\]
as claimed. □
Appendix II

In this appendix we provide a sufficient condition for the positivity of equilibrium prices. Let \( \zeta \in \mathbb{R}^{n+1} \) be a bundle. Denote the gradient of investor \( j \)'s utility function at \( \zeta \) by \( \nabla U_j^j(\zeta) = (U_0^j(\zeta), \ldots, U_n^j(\zeta)) \), where a subscript designates a partial derivative in the direction of the \( i \)th asset. Also, let \( \gamma_j^j(\zeta) = -\frac{V_{ij}^j(E(\tilde{y}_i), R(\zeta, \tilde{y}))}{V_i^j(E(\zeta, \tilde{y}), R(\zeta, \tilde{y}))} > 0 \) be the marginal rate of substitution of the expected payoff of the bundle for the risk of the bundle. This is the slope of investor \( j \)'s indifference curve in expected payoff-risk space. For brevity we often omit the arguments of this expression and use \( \gamma_j^j(\zeta) = -\frac{V_{ij}^j}{V_i^j} \).

Proposition 13 Assume that for each asset \( i \) there is some investor \( j \) such that \( E(\tilde{y}_i) > \gamma_j^j(\zeta) R_i(\zeta, \tilde{y}) \) for all \( \zeta \). Then, prices of all assets are positive in all equilibria.

Proof: As Nielsen (1992) indicates, at an equilibrium, all investors’ gradients point in the direction of the price vector. So the price of asset \( i \) must be positive in any equilibrium if there is some investor \( j \) such that \( U_i^j(\zeta) > 0 \) for all \( \zeta \). Recall that

\[
U_i^j(\zeta) = V_j^j(E(\zeta, \tilde{y}), R(\zeta, \tilde{y})).
\]

Thus,

\[
U_i^j(\zeta) = V_i^j E(\tilde{y}_i) + V_j^j R_i(\zeta, \tilde{y}) = V_i^j [E(\tilde{y}_i) - \gamma_j^j(\zeta) R_i(\zeta, \tilde{y})],
\]

where \( R_i(\zeta, \tilde{y}) \) denotes the partial derivative of \( R(\zeta, \tilde{y}) \) with respect to \( \zeta_i \).

Since \( V_i^j > 0 \), \( U_i^j(\zeta) > 0 \) corresponds to

\[
E(\tilde{y}_i) - \gamma_j^j(\zeta) R_i(\zeta, \tilde{y}) > 0
\]

as required. \( \blacksquare \)

Note that \( \gamma_j^j(\cdot) \) can serve as a measure of risk aversion for investor \( j \). We can thus interpret this proposition as follows. If each asset’s expected return is sufficiently high relative to some investor’s risk aversion and the marginal contribution of the asset to total risk, then this asset will always be desirable by some investor, and so, its price will be positive in any equilibrium.