Things I know, but sometimes forget

1 Every finite group acts freely on a product of spheres

It is not difficult to show that any finite group acts freely on a product of $S^3$'s. Indeed, $G$ acts freely on $\prod_{g \in G} i_g!S^3$, where $S^3$ is given a free $\langle g \rangle$-action and $i_g!S^3 = \text{map}_{\langle g \rangle}(G, S^3)$ is the co-induced $G$-space.

2 free $\times$ anything $= \text{free } \times \text{ anything else}$

If $X$ is a $G$-space, let $X_t$ be the same underlying space with the trivial $G$-action. Then there is a bijection of $G$-spaces $G \times X_t \to G \times X$ given by $(g, x) \mapsto (g, gx)$ with inverse map $(g, x) \mapsto (g, g^{-1}x)$. This can be generalized in two ways.

Let $i : H \to G$ be the inclusion of a subgroup. Let $i^* : G$-spaces $\to H$-spaces be the forgetful map given by restricting a $G$-action to an $H$-action and let $i_* : H$-spaces $\to G$-spaces be the induction map $i_*X = G \times_H X$. Then $i_*$ is the right adjoint of $i^*$. Let $X$ be a $G$-space. There is a homeomorphism

$$i_*i^*X = G \times_H i^*X \to G/H \times X$$

$$[g, x] \mapsto (gH, gx)$$

with inverse map $(gH, x) \mapsto [g, g^{-1}x]$.

If $F$ is a free $G$-set, choose a set of orbit representatives $B \subset F$ and define $F \times X_t \to F \times X$ by $(gb, x) \mapsto (gb, gx)$ for $b \in B$. 

1
3 Virtually cyclic groups come in three types

A virtually cyclic group is a group with a cyclic subgroup of finite index. They come in three types: finite, groups which surject to $\mathbb{Z}$ ($F \rtimes \mathbb{Z}$ with $F$ finite), and groups which surject to $D_\infty$ ($G_0 *_F G_1$ with $F$ finite and of index two).

**Theorem 1.** Let $\Gamma$ be an infinite virtually cyclic group.

1. If there is a central element of infinite order, then there is an epimorphism $\Gamma \to \mathbb{Z}$.

2. If there is not a central element of infinite order, then there is an epimorphism $\Gamma \to D_\infty$.

**Proof.** By intersecting the conjugates of an infinite cyclic subgroup, we may find a normal infinite cyclic subgroup $C$. Let $G$ be the finite quotient group.

1) In this case $G$ acts trivially on $C$. Embed $C$ as an index $|G|$ subgroup of an infinite cyclic group $C'$. Let $\Gamma' = C' \times_C \Gamma$. The image of the obstruction cocycle under the map $H^2(G;C) \to H^2(G;C')$ is trivial, so there exists a splitting $s : \Gamma' \to C'$ of the inclusion $C' \hookrightarrow \Gamma'$. Then $s|_{\Gamma} : \Gamma \to s(\Gamma)$ is the desired epimorphism.

2) Let $G_0 = \ker(G \to \text{Aut } C)$ (the map is by lifting to $\Gamma$ and using that conjugation preserves the normal subgroup.) Let $\Gamma_0 = \pi^{-1}G_0 < \Gamma$. Then there exists an epimorphism $\phi : \Gamma_0 \to \mathbb{Z}$ by 1). Likewise, $\Phi : \Gamma \to G \to G/G_0 \cong \mathbb{Z}_2$ is an epimorphism. Choose $\gamma \in \Gamma$ so that $\phi(\gamma) = 1$. Then $\Gamma = \Gamma_0 \amalg \Gamma_0 \gamma$. Define an epimorphism $\varphi : \Gamma \to \mathbb{Z} \times \mathbb{Z}_2$ by $\varphi(g) = (g,0)$ and $\varphi(g\gamma) = (g,1)$ for $g \in \Gamma_0$.

4 RAPL (= Right adjoints preserve limits)

Left adjoints preserve colimits, too! An *adjunction* is a pair of functors $C \xleftarrow{\phi} D$ and a natural isomorphism of functors

$$D^{\text{op}} \times C \to \text{Set}$$

$$D(d, U(c)) \cong C(F(d), c).$$
F is the left adjoint of U and U is the right adjoint of F.

Let \( I \) be a category and suppose \( C \) and \( D \) have \( I \)-limits. Let \( f : I \to C \) and \( g : I \to D \) be functors. Then the maps

\[
F(\operatorname{colim}_I f) \to \operatorname{colim}_I F \circ f \\
U(\operatorname{lim}_I g) \leftarrow \operatorname{lim}_I U \circ g
\]

are isomorphisms.

**Example 2.** Consider the adjunction \( \text{Set} \xleftarrow{F} \text{Group} \) with

\[
\text{Group}(F(X), G) \cong \text{Set}(X, U(G))
\]

where \( F \) takes a set to the free group generated by that set and \( U(G) \) is the forgetful functor taking a group to its underlying set. Let \( I \) be the category with two objects and only identity morphisms. Then

\[
F(X_1 \coprod X_1) \xrightarrow{\cong} F(X_1) * F(X_2) \\
U(G_1 \times G_2) \xleftarrow{\cong} U(G_1) \times U(G_2)
\]

**Example 3.** Let \( R \) be a ring and \( B \) be an \( R \)-module. Consider the adjunction

\[
- \otimes B : \text{R-mod} \to \text{R-mod} \\
\text{Hom}(B, -) : \text{R-mod} \to \text{R-mod} \\
\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))
\]

Let \( I \) be a category with only identity morphisms.

\[
(\bigoplus M_i) \otimes B \xrightarrow{\cong} \bigoplus (M_i \otimes B) \\
\text{Hom}(R, \prod M_i) \xleftarrow{\cong} \prod \text{Hom}(R, M_i)
\]

5 The degree of a cover equals the degree of a map

Let \( M \) be a closed, connected \( n \)-manifold. Then \( H_n M \) is zero or infinite cyclic. If \( H_n M \) is infinite cyclic, then we say \( M \) is orientable in which case \( H_n M \to H_n(M, M - \{x\}) \) is an isomorphism for all \( x \in M \).

There is an obvious local degree equals global degree proof of the following theorem, but this one, based on the transfer, is perhaps easier.
Theorem 4. Let \( p : \hat{M} \to M \) be a \( k \)-fold cover with domain and range closed, connected \( n \)-manifolds. If \( M \) is orientable then so is \( \hat{M} \) and \( p_* : H_n \hat{M} \to H_n M \) takes a generator to \( k \) times a generator.

Proof. We will define the transfer \( \text{tr} : H_i M \to H_i \hat{M} \) and show it is an isomorphism for \( i = n \). For a singular \( i \)-simplex \( \sigma : \Delta^i \to M \) there are exactly \( k \) singular \( i \)-simplices \( \bar{\sigma}^j : \Delta^j \to \hat{M} \), \( j = 1, \ldots, k \) so that \( p \circ \bar{\sigma}^j = \sigma \).

Define the chain map
\[
\text{tr}_\# : S_* M \to S_* \hat{M} \\
\sum a_\sigma \sigma \mapsto \sum a_\sigma \sum_{j=1}^k \bar{\sigma}^j.
\]
Clearly \( p_\# \circ \text{tr}_\# : S_* M \to S_* M \) is multiplication by \( k \) and the same is true after passing to homology. It follows that \( H_n \hat{M} \) is nonzero, hence \( \hat{M} \) is orientable.

Note that for a subset \( A \) of \( M \), the transfer map is also defined on relative homology \( \text{tr}_* : H_i(M, A) \to H_i(\hat{M}, p^{-1} A) \). Choose \( x \in M \) and \( y \in \hat{M} \) so that \( p(y) = x \). Consider the commutative diagram

\[
\begin{array}{ccc}
H_n(M, \hat{M} - \{y\}) & \xrightarrow{\sim} & H_n(\hat{M}, \hat{M} - y) \\
\text{tr}_* \downarrow & & \text{tr}_* \downarrow \\
H_n(M, M - \{x\}) & \xrightarrow{\sim} & H_n(M, M - x) \\
\end{array}
\]

If \( \sigma : \Delta^n \to M \) is an embedding with \( x \in \sigma(\text{int } \Delta^n) \), then by excision \( \sigma \) represents a generator of \( H_n(M, M - \{x\}) \). If, in addition, the image of \( \sigma \) is contained in a evenly covered neighborhood, then the images of the lifts \( \bar{\sigma}^1, \ldots, \bar{\sigma}^k \) are all disjoint, so \( y \) is contained in the image of exactly one of the lifts, say \( \bar{\sigma}^1 \). Then
\[
\pi_*(\text{tr}_*[\sigma]) = \pi_*[\bar{\sigma}^1 + \cdots + \bar{\sigma}^k] = [\bar{\sigma}^1]
\]
Hence the composite of the vertical maps on the right are isomorphisms, thus the transfer map on the left is an isomorphism. Since \( p_\ast \circ \text{tr}_\ast = k \cdot \text{Id} \), the result follows.

6 Inner automorphisms often induce identities

6.1 Groups

Recall a group is a category with one object.

**Lemma 5.** Let \( F : \text{Group} \to \text{Ab} \) be a functor. Suppose \( F(f) = F(g) \) for any natural transformation \( T : f \to g \) of morphisms of groups. Then for an inner automorphism \( c_\gamma : G \to G \) of a group, \( F(c_\gamma) = \text{Id}_{F(G)} \).

**Proof.** There is a natural transformation \( T : \text{Id}_G \to c_\gamma \) given by the morphism \( \gamma \).

**Corollary 6.** An inner automorphism induces the identity on the homology of a group.

**Proof.** Let \((0 \to 1)\) be the category with two objects and three morphisms, including a morphism from 0 to 1. A natural transformation \( T \) of functors \( F, F' : \mathcal{C} \to \mathcal{D} \) induces a functor \((0 \to 1) \times \mathcal{C} \to \mathcal{D}\) and conversely.

Let \( T : f \to g \) be a natural transformation of morphisms of groups \( f, g : G \to G' \). This induces a functor \((0 \to 1) \times G \to G'\) and hence a homotopy \( B(0 \to 1) \times BG \to BG' \) from \( Bf \) to \( Bg \).

Thus we can apply the Lemma above with \( F(G) = H_n(BG) \).

6.2 Rings

**Proposition 7.** An inner automorphism of a ring \( R \) induces the identity on \( K_nR \).

**Proof.** Let \( \gamma \in R^\times \). Consider the functor \( c_{\gamma^*} : \mathcal{P}(R) \to \mathcal{P}(R) \) given by \( c_{\gamma^*}(P) = R \otimes_{c_\gamma} P \). There is an exact natural transformation \( T : \text{Id} \to c_{\gamma^*} : \mathcal{P}(R) \to \mathcal{P}(R) \) given by \( P \to c_{\gamma^*}P \ x \mapsto \gamma^{-1}x \). It induces a functor

\[
(0 \to 1) \times Q(\mathcal{P}(R)) \to Q(\mathcal{P}(R))
\]

and hence a homotopy between the identity and \( BQ(c_{\gamma^*}) \).
7 A souped-up Hurewicz Theorem

A space $X$ is $n$-connected if every map $S^i \to X$ for $i \leq n$ is null-homotopic. The classical Hurewicz Theorem says that for an $n$-connected space, $\pi_i X \cong H_i X$ for $i \leq n + 1$.

**Theorem 8.** Let $k > 1$. If $X$ is $(k - 1)$-connected, the Hurewicz map $\pi_{k+1} X \to H_{k+1} X$ is onto.

The theorem is not true when $k = 1$. A counterexample is given by $S^1 \times S^1$.

**Proof.** First assume $X$ is an Eilenberg-MacLane space $K(G, k)$ with $G$ and abelian group and $k > 1$. There is a short exact sequence of abelian groups

$$0 \to F' \to F \to G \to 0$$

where $F$ and $F'$ are free abelian groups. (Indeed, find a surjection $\phi : F \to G$ with $F$ a free abelian group and note that the subgroup $\ker \phi < F$ is itself free abelian.) By choosing bases for $F$ and $F'$, build a CW complex $Y$ with only a 0-cell, $k$-cells, and $(k + 1)$-cells, with $\pi_k Y = G$, and with $H_{k+1} Y = 0$. Build a $K(G, k)$ by adding on cells of dimension $k + 2$ and higher. Then $H_{k+1} K(G, k)$ is a quotient of $\ker(\partial : C_{k+1} Y \to C_k Y) = \ker(F' \to F) = 0$.

Now we prove the theorem for a general $(k - 1)$-connected space $X$ where $k > 1$. Let $G = \pi_k X$. Choose a map $X \to K(G, k)$ which is the identity on $\pi_k$. Let $F$ be the homotopy fiber. Then the Serre exact sequence (which follows from the Serre spectral sequence) gives a long exact sequence

$$H_{2k} F \to H_{2k} X \to H_{2k} K(G, k) \to \cdots$$

$$\to H_{k+1} F \to H_{k+1} X \to H_{k+1} K(G, k) \to \cdots$$

There is a commutative diagram

$$\begin{array}{ccc}
\pi_{k+1} F & \xrightarrow{\sim} & H_{k+1} F \\
\downarrow & & \downarrow \\
\pi_{k+1} X & \longrightarrow & H_{k+1} X
\end{array}$$

The left map is an isomorphism because of the homotopy exact sequence and the top map is an isomorphism by the Hurewicz Theorem. Since $H_{k+1} K(G, k) = 0$ by our above arguments, the Serre exact sequence shows the right hand map is onto. Thus the bottom map is onto as desired. $\square$
8 Poincaré duality and local coefficients

Let $X$ be a connected CW-complex with fundamental group $\pi$ and universal cover $\tilde{X}$. If $A$ is a left (right) $\mathbb{Z}\pi$-module, let $\overline{A}$ be the the right (left) $\mathbb{Z}\pi$-module defined by $a\lambda = \overline{\lambda}a$ ($\lambda a = a\overline{\lambda}$) where $\sum a_g g = \sum a_g g^{-1}$. Let $\tilde{X} \curvearrowright \pi$ be the right action by deck transformations. For a left $\mathbb{Z}\pi$-module $A$, let $H_*(X; A)$ be the homology of the chain complex $C_*(X; A) = C_\ast\tilde{X} \otimes \mathbb{Z}\pi A$. For a right $\mathbb{Z}\pi$-module $A$, let $H^*(X; A)$ be the cohomology of the cochain complex $C^*(X; A) = \text{Hom}_{\mathbb{Z}\pi}(C_\ast\tilde{X}, A)$. Note that $H_0(X; A) = A_G$ (invariants) and that $H_0(X; A) = A_G = A \otimes_{\mathbb{Z}G} \mathbb{Z}$ (coinvariants). Cup and cap products with local coefficients work as expected:

$$H^i(X; A) \times H^j(X; B) \to H^{i+j}(X; A \otimes \mathbb{Z} B)$$

$$H^i(X; A) \times H^j(X; B) \to H_{i-j}(X; \overline{A} \otimes \mathbb{Z} B)$$

where we take the diagonal right $\pi$-action on $A \otimes \mathbb{Z} B$ and the diagonal left $\pi$-action on $\overline{A} \otimes B$.

Then Poincaré duality states:

**Theorem 9.** Let $X$ be a closed, connected, oriented $n$-manifold, $[X] \in H_n(X; \mathbb{Z})$ the fundamental class and $A$ any right $\mathbb{Z}\pi$-module. Then

$$\cap [X] : H^{n-i}(X; A) \xrightarrow{\cong} H_i(X; \overline{A})$$

The same sort of thing is true for nonorientable manifolds; let $[X] \in H_n(X; \mathbb{Z}_w)$ be a generator; then

$$\cap [X] : H^{n-i}(X; A) \xrightarrow{\cong} H_i(X; \overline{A}_w)$$

**Remark 10.** Poincaré duality for $A$ is a formal consequence of Poincaré duality with $\mathbb{Z}\pi$-coefficients. Indeed, $\cap [X] : C^{n-\ast}(X; \mathbb{Z}\pi) \to C_\ast(X; \mathbb{Z}\pi)$ is a chain homotopy equivalence, hence so is $\cap [X] : C^{n-\ast}(X; \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} \overline{A} \to C_\ast(X; \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} \overline{A}$. But this can be identified with $\cap [X] : C^{n-\ast}(X; A) \to C_\ast(X; \overline{A})$.

**Remark 11.** This gives intersection pairings. Indeed, let $A$ and $B$ be right $\mathbb{Z}\pi$-modules. Note that $(A \otimes \mathbb{Z} B)_\pi = A \otimes_{\mathbb{Z}G} \overline{B}$ with $a \otimes b \leftrightarrow a \otimes \overline{b}$. Then by applying the cup product, then Poincaré duality, the computation of $H_0$, and the above identification, one has a bilinear pairing for $i + j = n$,

$$H^i(X; A) \times H^j(X; B) \to A \otimes_{\mathbb{Z}\pi} \overline{B}$$
If \( \mathbb{Z} \pi \to R \) is an epimorphism of rings with involution (e.g. the identity), and if \( A = B = R \), there are isomorphisms of \( \mathbb{Z} \pi - \mathbb{Z} \pi \) bimodules

\[
R \otimes_{\mathbb{Z} \pi} \overline{R} = \mathbb{R} \otimes_R \overline{R} \to R
\]

\[
\alpha \otimes \beta \mapsto \alpha \overline{\beta}
\]

This gives an \( R \)-valued intersection pairing.