Things I know, but sometimes forget

1 Every finite group acts freely on a product of spheres

It is not difficult to show that any finite group acts freely on a product of $S^3$'s. Indeed, $G$ acts freely on $\prod_{g \in G} i_g S^3$, where $S^3$ is given a free $(g)$-action and $i_g S^3 = \text{map}_<(g)(G, S^3)$ is the co-induced $G$-space.

2 free $\times$ anything $=$ free $\times$ anything else

If $X$ is a $G$-space, let $X_t$ be the same underlying space with the trivial $G$-action. Then there is a bijection of $G$-spaces $G \times X_t \to G \times X$ given by $(g, x) \mapsto (g, gx)$ with inverse map $(g, x) \mapsto (g, g^{-1}x)$. This can be generalized in two ways.

Let $i : H \to G$ be the inclusion of a subgroup. Let $i^* : G$-spaces $\to H$-spaces be the forgetful map given by restricting a $G$-action to an $H$-action and let $i_* : H$-spaces $\to G$-spaces be the induction map $i_* X = G \times_H X$. Then $i_*$ is the right adjoint of $i^*$. Let $X$ be a $G$-space. There is a homeomorphism

$$i_* i^* X = G \times_H i^* X \to G/H \times X$$

$$[g, x] \mapsto (gH, gx)$$

with inverse map $(gH, x) \mapsto [g, g^{-1}x]$.

If $F$ is a free $G$-set, choose a set of orbit representatives $B \subset F$ and define $F \times X_t \to F \times X$ by $(gb, x) \mapsto (gb, gx)$ for $b \in B$. 
3 Virtually cyclic groups come in three types

A virtually cyclic group is a group with a cyclic subgroup of finite index. They come in three types: finite, groups which surject to \( \mathbb{Z} \) (\( F \rtimes \mathbb{Z} \) with \( F \) finite), and groups which surject to \( D_\infty \) (\( G_0 \ast F \) \( G_1 \) with \( F \) finite and of index two).

**Theorem 1.** Let \( \Gamma \) be an infinite virtually cyclic group.

1. If there is a central element of infinite order, then there is an epimorphism \( \Gamma \to \mathbb{Z} \).

2. If there is not a central element of infinite order, then there is an epimorphism \( \Gamma \to D_\infty \).

**Proof.** By intersecting the conjugates of an infinite cyclic subgroup, we may find an normal infinite cyclic subgroup \( C \). Let \( G \) be the finite quotient group.

1) In this case \( G \) acts trivially on \( C \). Embed \( C \) as an index \( |G| \) subgroup of an infinite cyclic group \( C' \). Let \( \Gamma' = C' \times_C \Gamma \). The image of the obstruction cocycle under the map \( H^2(G; C) \to H^2(G; C') \) is trivial, so there exists a splitting \( s : \Gamma' \to C' \) of the inclusion \( C' \hookrightarrow \Gamma' \). Then \( s|_{\Gamma} : \Gamma \to s(\Gamma) \) is the desired epimorphism.

2) Let \( G_0 = \ker(G \to \text{Aut } C) \) (the map is by lifting to \( \Gamma \) and using that conjugation preserves the normal subgroup.) Let \( \Gamma_0 = \pi^{-1}G_0 < \Gamma \). Then there exists an epimorphism \( \phi : \Gamma_0 \to \mathbb{Z} \) by 1). Likewise, \( \Phi : \Gamma \to G \to G/G_0 \cong \mathbb{Z}_2 \) is an epimorphism. Choose \( \gamma \in \Gamma \) so that \( \phi(\gamma) = 1 \). Then \( \Gamma = \Gamma_0 \ast \Gamma_0 \gamma \). Define a epimorphism \( \varphi : \Gamma \to \mathbb{Z} \times \mathbb{Z}_2 \) by \( \varphi(g) = (g,0) \) and \( \varphi(g\gamma) = (g,1) \) for \( g \in \Gamma_0 \).

4 RAPL (= Right adjoints preserve limits)

Left adjoints preserve colimits, too! An adjunction is a pair of functors \( C \xleftarrow{\Phi} D \) and a natural isomorphism of functors

\[
\begin{align*}
D^{\text{op}} \times C & \to \text{Set} \\
D(d, U(c)) & \cong C(F(d), c).
\end{align*}
\]
$F$ is the left adjoint of $U$ and $U$ is the right adjoint of $F$.

Let $\mathcal{I}$ be a category and suppose $\mathcal{C}$ and $\mathcal{D}$ have $\mathcal{I}$-limits. Let $f : \mathcal{I} \to \mathcal{C}$ and $g : \mathcal{I} \to \mathcal{D}$ be functors. Then the maps

$$ F(\operatorname{colim} f) \to \operatorname{colim} F \circ f $$
$$ U(\operatorname{lim} g) \leftarrow \operatorname{lim} U \circ g $$

are isomorphisms.

**Example 2.** Consider the adjunction $\operatorname{Set} \overset{F}{\leftrightarrow} \operatorname{Group}$ with

$$ \operatorname{Group}(F(X), G) \cong \operatorname{Set}(X, U(G)) $$

where $F$ takes a set to the free group generated by that set and $U(G)$ is the forgetful functor taking a group to its underlying set. Let $\mathcal{I}$ be the category with two objects and only identity morphisms. Then

$$ F(X_1 \coprod X_1) \cong F(X_1) \ast F(X_2) $$
$$ U(G_1 \times G_2) \cong U(G_1) \times U(G_2) $$

**Example 3.** Let $R$ be a ring and $B$ be an $R$-module. Consider the adjunction

$$ - \otimes B : \operatorname{R-mod} \to \operatorname{R-mod} $$
$$ \operatorname{Hom}(B, -) : \operatorname{R-mod} \to \operatorname{R-mod} $$
$$ \operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C)) $$

Let $\mathcal{I}$ be a category with only identity morphisms.

$$ (\bigoplus M_i) \otimes B \cong \bigoplus (M_i \otimes B) $$
$$ \operatorname{Hom}(R, \prod M_i) \cong \prod \operatorname{Hom}(R, M_i) $$

5 The degree of a cover equals the degree of a map

Let $M$ be a closed, connected $n$-manifold. Then $H_n M$ is zero or infinite cyclic. If $H_n M$ is infinite cyclic, then we say $M$ is orientable in which case $H_n M \to H_n(M, M - \{x\})$ is an isomorphism for all $x \in M$.

There is an obvious local degree equals global degree proof of the following theorem, but this one, based on the transfer, is perhaps easier.
Theorem 4. Let $p: \hat{M} \to M$ be a $k$-fold cover with domain and range closed, connected $n$-manifolds. If $M$ is orientable then so is $\hat{M}$ and $p_* : H_n \hat{M} \to H_n M$ takes a generator to $k$ times a generator.

Proof. We will define the transfer $\text{tr} : H_i M \to H_i \hat{M}$ and show it is an isomorphism for $i = n$. For a singular $i$-simplex $\sigma : \Delta^i \to M$ there are exactly $k$ singular $i$-simplices $\bar{\sigma}^j : \Delta^j \to \hat{M}$, $j = 1, \ldots, k$ so that $p \circ \bar{\sigma}^j = \sigma$. Define the chain map

$$\text{tr}_\# : S_* M \to S_* \hat{M}$$

$$\sum a_\sigma \sigma \mapsto \sum a_\sigma \sum_{j=1}^k \bar{\sigma}^j.$$

Clearly $p_\# \circ \text{tr}_\# : S_* M \to S_* M$ is multiplication by $k$ and the same is true after passing to homology. It follows that $H_n \hat{M}$ is nonzero, hence $\hat{M}$ is orientable.

Note that for a subset $A$ of $M$, the transfer map is also defined on relative homology $\text{tr}_* : H_i(M, A) \to H_i(\hat{M}, p^{-1}A)$. Choose $x \in M$ and $y \in \hat{M}$ so that $p(y) = x$. Consider the commutative diagram

\[
\begin{array}{ccc}
H_n(M, \hat{M} - \{y\}) & \overset{\approx}{\longrightarrow} & H_n(M, M - \{x\}) \\
\downarrow \text{tr}_* & & \downarrow \text{tr}_* \\
H_n(\hat{M}, \hat{M} - p^{-1}\{x\}) & \overset{\approx}{\longrightarrow} & H_n(M, M - \{x\})
\end{array}
\]

If $\sigma : \Delta^n \to M$ is an embedding with $x \in \sigma(\text{int } \Delta^n)$, then by excision $\sigma$ represents a generator of $H_n(M, M - \{x\})$. If, in addition, the image of $\sigma$ is contained in an evenly covered neighborhood, then the images of the lifts $\bar{\sigma}^1, \ldots, \bar{\sigma}^k$ are all disjoint, so $y$ is contained in the image of exactly one of the lifts, say $\bar{\sigma}^1$. Then

$$\pi_*(\text{tr}_*[\sigma]) = \pi_*[\bar{\sigma}^1 + \cdots + \bar{\sigma}^k] = [\bar{\sigma}^1]$$
Hence the composite of the vertical maps on the right are isomorphisms, thus the transfer map on the left is an isomorphism. Since \( p_* \circ \text{tr}_* = k \cdot \text{Id} \), the result follows.

6 Inner automorphisms often induce identities

6.1 Groups

Recall a group is a category with one object.

**Lemma 5.** Let \( F : \text{Group} \rightarrow \text{Ab} \) be a functor. Suppose \( F(f) = F(g) \) for any natural transformation \( T : f \rightarrow g \) of morphisms of groups. Then for an inner automorphism \( c_\gamma : G \rightarrow G \) of a group, \( F(c_\gamma) = \text{Id}_{F(G)} \).

**Proof.** There is a natural transformation \( T : \text{Id}_G \rightarrow c_\gamma \) given by the morphism \( \gamma \).

**Corollary 6.** An inner automorphism induces the identity on the homology of a group.

**Proof.** Let \( (0 \rightarrow 1) \) be the category with two objects and three morphisms, including a morphism from 0 to 1. A natural transformation \( T \) of functors \( F, F' : C \rightarrow D \) induces a functor \( (0 \rightarrow 1) \times C \rightarrow D \) and conversely.

Let \( T : f \rightarrow g \) be a natural transformation of morphisms of groups \( f,g : G \rightarrow G' \). This induces a functor \( (0 \rightarrow 1) \times G \rightarrow G' \) and hence a homotopy \( B(0 \rightarrow 1) \times BG \rightarrow BG' \) from \( Bf \) to \(Bg\).

Thus we can apply the Lemma above with \( F(G) = H_n(BG) \).

6.2 Rings

**Proposition 7.** An inner automorphism of a ring \( R \) induces the identity on \( K_n R \).

**Proof.** Let \( \gamma \in R^\times \). Consider the functor \( c_{\gamma*} : \mathcal{P}(R) \rightarrow \mathcal{P}(R) \) given by \( c_{\gamma*}(P) = R \otimes_{c_\gamma} P \). There is an exact natural transformation \( T : \text{Id} \rightarrow c_{\gamma*} : \mathcal{P}(R) \rightarrow \mathcal{P}(R) \) given by \( P \rightarrow c_{\gamma*}P \ x \mapsto \gamma^{-1}x \). It induces a functor \( (0 \rightarrow 1) \times Q(\mathcal{P}(R)) \rightarrow Q(\mathcal{P}(R)) \) and hence a homotopy between the identity and \( BQ(c_{\gamma*}) \).
7 A souped-up Hurewicz Theorem

A space \(X\) is \(n\)-connected if every map \(S^i \to X\) for \(i \leq n\) is null-homotopic. The classical Hurewicz Theorem says that for an \(n\)-connected space, \(\pi_* X \cong H_* X\) for \(i \leq n + 1\).

**Theorem 8.** Let \(k > 1\). If \(X\) is \((k - 1)\)-connected, the Hurewicz map \(\pi_{k+1} X \to H_{k+1} X\) is onto.

The theorem is not true when \(k = 1\). A counterexample is given by \(S^1 \times S^1\).

**Proof.** First assume \(X\) is an Eilenberg-MacLane space \(K(G,k)\) with \(G\) and abelian group and \(k > 1\). There is a short exact sequence of abelian groups

\[
0 \to F' \to F \to G \to 0
\]

where \(F\) and \(F'\) are free abelian groups. (Indeed, find a surjection \(\phi : F \to G\) with \(F\) a free abelian group and note that the subgroup \(\ker \phi < F\) is itself free abelian.) By choosing bases for \(F\) and \(F'\), build a CW complex \(Y\) with only a 0-cell, \(k\)-cells, and \((k + 1)\)-cells, with \(\pi_k Y = G\), and with \(H_{k+1} Y = 0\). Build a \(K(G,k)\) by adding on cells of dimension \(k + 2\) and higher. Then \(H_{k+1} K(G,k)\) is a quotient of \(\ker(\partial : C_{k+1} Y \to C_k Y) = \ker(F' \to F) = 0\).

Now we prove the theorem for a general \((k - 1)\)-connected space \(X\) where \(k > 1\). Let \(G = \pi_k X\). Choose a map \(X \to K(G,k)\) which is the identity on \(\pi_k\). Let \(F\) be the homotopy fiber. Then the Serre exact sequence (which follows from the Serre spectral sequence) gives a long exact sequence

\[
H_{2k} F \to H_{2k} X \to H_{2k} K(G,k) \to \cdots
\]

\[
\to H_{k+1} F \to H_{k+1} X \to H_{k+1} K(G,k) \to \cdots
\]

There is a commutative diagram

\[
\begin{array}{ccc}
\pi_{k+1} F & \xrightarrow{\sim} & H_{k+1} F \\
\downarrow & & \downarrow \\
\pi_{k+1} X & \longrightarrow & H_{k+1} X
\end{array}
\]

The left map is an isomorphism because of the homotopy exact sequence and the top map is an isomorphism by the Hurewicz Theorem. Since \(H_{k+1} K(G,k) = 0\) by our above arguments, the Serre exact sequence shows the right hand map is onto. Thus the bottom map is onto as desired. \(\Box\)
8 Poincaré duality and local coefficients

Let $X$ be a connected CW-complex with fundamental group $\pi$. If $A$ is a left (right) $\mathbb{Z}\pi$-module, let $\overline{A}$ be the the right (left) $\mathbb{Z}\pi$-module defined by $a\lambda = \overline{\lambda}a$ ($\lambda = a\overline{\lambda}$) where $\sum a_g g = \sum a_g g^{-1}$. Let $\tilde{X} \curvearrowright \pi$ be the right action by deck transformations. For a left $\mathbb{Z}\pi$-module $A$, let $H_*(X; A)$ be the homology of the chain complex $C_*(X; A) = C_*\tilde{X} \otimes_{\mathbb{Z}\pi} A$. For a right $\mathbb{Z}\pi$-module $A$, let $H^*(X; A)$ be the cohomology of the cochain complex $C^*(X; A) = \text{hom}_{\mathbb{Z}\pi}(C_*\tilde{X}, A)$. Note that $H^0(X; A) = A_G$ and that $H_0(X; A) = A_G$ (coinvariants). Cup and cap products with local coefficients work as expected:

$$H^i(X; A) \times H^j(X; B) \to H^{i+j}(X; A \otimes_{\mathbb{Z}} B)$$

where we take the diagonal right $\pi$-action on $A \otimes_{\mathbb{Z}} B$ and the diagonal left $\pi$-action on $A \otimes B$.

Then Poincaré duality states:

**Theorem 9.** Let $X$ be a closed, connected, oriented $n$-manifold, $[X] \in H_n(X; \mathbb{Z})$ the fundamental class and $A$ any right $\mathbb{Z}\pi$-module. Then

$$\cap[X] : H^{n-i}(X; A) \xrightarrow{\cong} H_i(X; \overline{A})$$

The same sort of thing is true for nonorientable manifolds; let $[X] \in H_n(X; \mathbb{Z}_w)$ be a generator, then

$$\cap[X] : H^{n-i}(X; A) \xrightarrow{\cong} H_i(X; \overline{A}_w)$$

**Remark 10.** Poincaré duality for $A$ is a formal consequence of Poincaré duality with $\mathbb{Z}\pi$-coefficients. Indeed, $\cap[X] : C^{n-*}(X; \mathbb{Z}_\pi) \to C_*(X; \mathbb{Z}_\pi)$ is a chain homotopy equivalence, hence so is $\cap[X] : C^{n-*}(X; \mathbb{Z}_\pi) \otimes_{\mathbb{Z}_\pi} A \to C_*(X; \mathbb{Z}_\pi) \otimes_{\mathbb{Z}_\pi} A$. But this equals $\cap[X] : C^{n-*}(X; A) \to C_*(X; \overline{A})$.

**Remark 11.** This gives intersection pairings. Indeed, let $A$ and $B$ be right $\mathbb{Z}\pi$-modules. Note that $(A \otimes_{\mathbb{Z}} B)_{\pi} = A \otimes_{\mathbb{Z}_\pi} \overline{B}$ with $a \otimes b \leftrightarrow a \otimes \overline{b}$. Then by applying the cup product, then Poincaré duality, the computation of $H_0$, and the above identification, one sees that for $i + j = n$,

$$H^i(X; A) \times H^j(X; B) \to A \otimes_{\mathbb{Z}_\pi} \overline{B}$$
In the special case where $A = B = \mathbb{Z}_\pi$, there is an isomorphism of $\mathbb{Z}_\pi - \mathbb{Z}_\pi$ bimodules

$$\mathbb{Z}_\pi \otimes_{\mathbb{Z}_\pi} \mathbb{Z}_\pi \to \mathbb{Z}_\pi$$

$$\alpha \otimes \beta \mapsto \alpha \beta$$

This gives a $\mathbb{Z}_\pi$-valued intersection pairing.