The study of vector bundles is the study of parameterized linear algebra.

**Definition 1.** A *vector bundle* is a map $\pi : E \to B$ together with a vector space structure on $\pi^{-1}b$ for each $b \in B$ so that for every $x \in B$ there is a neighborhood $U$ and a $k \in \mathbb{Z}_{\geq 0}$, a homeomorphism $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ so that there is a commutative diagram

$$
\begin{array}{ccc}
\pi^{-1}U & \xrightarrow{\phi} & U \times \mathbb{R}^k \\
\downarrow & \cong & \downarrow \\
U & \to & 
\end{array}
$$

so that the induced bijection $\pi^{-1}b \to \{b\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a vector space isomorphism for all $b \in U$.

$B$ is called the *base space*, $E$ is called the *total space*, and the vector spaces $E_b = \pi^{-1}b$ are called the *fibers*.

A vector bundle is *smooth* if $E$ and $B$ are smooth manifolds, $\pi$ is a smooth map and if for every $x \in B$ there is a neighborhood $U$ and a smooth chart $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ as above.

A *map of vector bundles* is a commutative diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
B' & \xrightarrow{f} & B
\end{array}
$$
which induces a linear map on the “fibers” $\pi^{-1}x \rightarrow \pi'^{-1}f(x)$.

An example of a vector bundle is the tangent bundle of a manifold. The differential of a smooth map $f : X \rightarrow Y$ gives a map of vector bundles $df : TX \rightarrow TY$.

If $X \subset \mathbb{R}^n$ then

$$TX = \{(p, v) \in X \times \mathbb{R}^n \mid v \text{ is the tangent vector of a curve in } X \text{ through } p\}$$

If $X^k$ is an abstract smooth manifold with atlas $\mathcal{A} = \{\phi : V \rightarrow U \subset \mathbb{R}^k\}$ then the tangent bundle can be defined as a quotient

$$TX = \bigsqcup_{U} U \times \mathbb{R}^k \sim$$

A map of vector bundles over $B$ is a commutative diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{j} & E \\
\downarrow & & \downarrow \\
B & & B
\end{array}
\]

which induces a linear map on the fibers.

2 Extra structure on vector bundles

Definition 2. An oriented vector bundle is a vector bundle $\pi : E \rightarrow B$ together with an orientation on each fiber, so that there is an atlas of charts $\{\phi_U : \pi^{-1}U \rightarrow U \times \mathbb{R}^k\}$ inducing orientation-preserving isomorphisms $\pi^{-1}b \rightarrow \mathbb{R}^k$ for each chart $\phi_U$ and for each $b \in U$.

An oriented manifold is a manifold $X$ with an orientation on its tangent bundle $TX$.

Definition 3. An vector bundle with metric is a vector bundle $\pi : E \rightarrow B$ together with an inner product $\langle \cdot , \cdot \rangle_b : \pi^{-1}b \times \pi^{-1}b \rightarrow \mathbb{R}$ on each fiber so that there is an atlas of charts $\{\phi_U : \pi^{-1}U \rightarrow U \times \mathbb{R}^k\}$ inducing isometries for each chart $\phi_U$ and for each $b \in U$.

Every vector bundle over a paracompact space admits a metric.

An Riemannian manifold is a manifold $X$ with a smooth metric on its tangent bundle $TX$. 

2
3 New vector bundles from old

Definition 4. Given vector bundles $\pi' : E' \to B'$ and $\pi : E \to B$, the product bundle is product map $\pi' \times \pi : E' \times E \to B' \times B$.

Definition 5. Given vector bundles $\pi' : E' \to B$ and $\pi : E \to B$, the Whitney sum is the bundle $E' \oplus E \to B$ where $E' \oplus E = \{ (e', e) \in E' \times E \mid \pi'(e') = \pi(e) \}$. The fiber above $b$ is $\pi'^{-1}b' \oplus \pi^{-1}b$.

Definition 6. A subbundle of a bundle $\pi : E \to B$ is a subspace $E' \subset E$ so that $\pi|_{E'} : E' \to B$ is a vector bundle. Given a subbundle, there is the quotient bundle $E/\sim_{E'} \to B$ where $\sim_{E'}$ is the equivalence relation on $E$ given by $e_1 \sim e_2$ if they are both in the same fiber and if $e_1 - e_2 \in E'$.

If $E' \to B$ is a subbundle of a bundle $E \to B$ with a metric, then $E \to B$ is a Whitney sum $E' \oplus E'^\perp \to B$, where $E'^\perp = \{ e \in E \mid \langle e, E'_{\pi(b)} \rangle = 0 \}$. Furthermore the obvious map $E'^\perp \to E/\sim_{E'}$ gives an isomorphism of vector bundles over $B$.

As a consequence one sees that a short exact sequence $$0 \to E' \to E \to E'' \to 0$$ of vector bundles over a paracompact $B$ splits.

The restriction of a vector bundle $\pi : E \to B$ to $B' \subset B$ is the vector bundle $\pi^{-1}B' \to B'$. We write this as $E|_{B'} \to B'$.

Example 7. Let $X^k \subset Y^l \subset \mathbb{R}^n$ be submanifolds. Let $N(X \subset Y)$ be the orthogonal complement $TX^\perp$ of $TX$ in $TY|_X$. Let $\nu(X \subset Y)$ be the quotient bundle $(TY|_X)/TX$ (or rather $TY|_X/\sim_{TX}$ in the previous notation). We call both of these (isomorphic) bundles the normal bundle of $X \subset Y$. Note $TY|_X = TX \oplus TX^\perp = TX \oplus N(X \subset Y)$

In particular the tangent bundle and normal bundle of $X \subset \mathbb{R}^n$ are Whitney sum inverses.

Definition 8. Given a vector bundle $\pi : E \to B$ and a map $f : B' \to B$, the pullback bundle is given by $f^*E \to B$ where $f^*E = \{ (b', e) \in B' \times E \mid f(b') = \pi(e) \}$. (One also writes $f^*E = B' \times_B E$.) Use the commutative diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{\pi_2} & E \\
\downarrow{\pi_1} & & \downarrow{\pi} \\
B' & \xrightarrow{f} & B
\end{array}
$$
which induces a bijection on the fibers to define the vector space structure on the pullback.

As an example, if \( i : B' \hookrightarrow B \) is the inclusion then \( E|_{B'} \) is the pullback bundle \( i^*E \).

**Exercise 9.** Suppose

\[
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow \pi' & & \downarrow \pi \\
B' & \overset{f}{\longrightarrow} & B
\end{array}
\]

\( \pi \) and \( \pi' \) are vector bundles and \( f \) is a continuous map. There is a bijection between vector bundle maps \( \tilde{f} : E' \to E \) over \( f \) and vector bundle maps \( E' \to f^*E \) over \( B \). In particular, there is a fiberwise isomorphism covering \( f \) if and only if \( E' \) and \( f^*E \) are isomorphic vector bundles over \( B \).

\[
\begin{array}{ccc}
E' \longrightarrow & f^*E & \longrightarrow E \\
\downarrow \pi' & & \downarrow \pi \\
B' & \overset{f}{\longrightarrow} & B
\end{array}
\]

4 **Bundles and transversality**

**Lemma 10.** Let \( f : X \to Y \) be a linear transformation, \( Z \subset Y \) be a subspace, and \( S = f^{-1}Z \). Then

\[ f(X) + Z = Y \iff \overline{f} : X/S \to Y/Z \text{ is an isomorphism.} \]

**Theorem 11.** Let \( f : X \to Y \) be smooth map of manifolds and \( Z \subset Y \) be a submanifold. Then

\[ f \pitchfork Z \iff \]

1. \( S = f^{-1}Z \) is a manifold.

2. \( df : \nu(S \subset X) \to \nu(Z \subset Y) \) is a fiberwise isomorphism of vector bundles (\( \iff df : \nu(S \subset X) \cong f^*\nu(Z \subset Y) \)).
5 Bundles, orientation, and transversality

An orientation on two of the three vector spaces $E', E'', \text{ and } E' \oplus E''$ determines a orientation on the third. The same is true with vector spaces replaced by vector bundles over $B$.

Given a short exact sequence of vectors spaces

$$0 \to E' \to E \to E'' \to 0$$

an orientation on two of the three vector spaces determines an orientation on the third. The same is true with vector spaces replaced by vector bundles over $B$.

**Definition 12.** Suppose $f : X \to Y$ with $f \pitchfork Z$. Suppose $X, Y, \text{ and } Z$ are oriented. Then we orient $S = f^{-1}Z$ (equivalently we oriented $TS$) using the equations

1. $N(Z \subset Y) \oplus TZ = TY|_Z$
2. $df : N(S \subset X) \xrightarrow{\cong} f^*N(Z \subset Y)$
3. $N(S \subset X) \oplus TS = TX|_S$

Note that (unfortunately) order matters in points 1 and 3 above.

**References**
