Elliptic Regularity

Throughout we assume all vector bundles are smooth bundles with metrics over a Riemannian manifold $X^n$.

1 Review of Hodge Theory

In this note I outline the proof of the following Fundamental Theorem.

**Theorem 1** (Fundamental Theorem). Let $\Delta : C^\infty(E) \to C^\infty(E)$ be a self-adjoint elliptic differential operator. Then

- $\dim \ker \Delta < \infty$
- $C^\infty(E) = \ker \Delta \oplus \text{im} \Delta$

There are many, many corollaries to this. However, they can all be subsumed into the following.

**Corollary 2.** Let $(C^\infty(E^\bullet),D))$ be an elliptic complex. Let $\Delta = \Delta^p = DD^* + D^*D : C^\infty(E^p) \to C^\infty(E^p)$. Then $(C^\infty(E^\bullet),D)) = \ker \Delta \oplus \text{im} \Delta$ where the maps labeled $D$ are isomorphisms.
Corollary 3 (Hodge Theorem). \( \ker \Delta^p \cong H^p(C^\infty(E^*), D) \)

Corollary 4. For an elliptic differential operator \( \ker D^* \to \text{cok} D \) is an isomorphism.

For more details see the algebraic Hodge theory post at wordpress.indextheory.com

Classically, one considers the exterior derivative. Then an harmonic form is a form in the kernel of the Laplacian. The Hodge theorem says that every DeRham cohomology class has a unique harmonic representative. Furthermore this harmonic representative minimizes the \( L^2 \)-norm within the cohomology class (since \( C^\infty E^p = \ker \Delta \oplus \text{im} D \oplus \text{im} D^* \)).

2 Weak Solutions

Here are some ideas behind the proof. Let \( \mathcal{H} \subset C^\infty E \) be the kernel of the Laplacian \( \Delta \). Note \( \text{im} \Delta \subset \mathcal{H}^\perp \)

One needs to show the opposite inclusion. So in other words for any \( t_0 \in \mathcal{H}^\perp \), one needs to find a solution to the equation \( \Delta(\cdot) = t_0 \). This leads one to think about Hilbert space methods and the Riesz Representation Theorem.

Often our discussion will be clearer if we consider a non-necessarily self-adjoint differential operator \( D : C^\infty(E) \to C^\infty(F) \).

Given \( t_0 \in C^\infty(F) \), a weak solution to \( D(\cdot) = t_0 \) is a bounded linear functional \( l : C^\infty(E) \to \mathbb{C} \) so that \( l(D^*t) = \langle t, t_0 \rangle \) \( \forall t \in C^\infty(F) \).

The following theorem is proved in Warner:

**Theorem 5.** Let \( D : C^\infty(E) \to C^\infty(F) \) be an elliptic differential operator. Then any weak solution \( l \) to \( D(\cdot) = t_0 \) is represented by a smooth solution \( s_0 \in C^\infty(E) \), i.e. \( l(\cdot) = \langle -, s_0 \rangle \) which implies \( Ds_0 = t_0 \).

Note that \( C^\infty(E) \) and \( C^\infty(F) \) are inner product spaces so can be completed to Hilbert spaces \( L^2(E) \) and \( L^2(F) \). But one cannot extend \( D \) to these Hilbert spaces since one cannot differentiate measurable sections. This leads us to:
3 Sobolev Spaces

Throughout we will assume our vector bundles have covariant derivatives.

Definition 6. For an integer \( k \geq 0 \), define an inner product on \( C^\infty(E) \) by

\[
\|s\|_k = \sqrt{\sum_{j=0}^{k} \int_X \|\nabla^j s\|^2}
\]

where \( \nabla^2 \) is, for instance, the composite

\[
C^\infty(E) \xrightarrow{\nabla^E} C^\infty(T^*X \otimes E) \xrightarrow{\nabla^{T^*X \otimes E}} C^\infty(T^*X \otimes T^*X \otimes E)
\]

The Sobolev space \( L_k(E) \) is the completion of \( C^\infty(E) \) under this norm. Define \( L_{-k}(E) = L_k(E)^* \).

Proposition 7. A differential operator of order \( m \) induces a bounded linear map

\[
D_k : L_k(E) \to L_{k-m}(F).
\]

Recall bounded means there is a \( C \) so that \( \|Ds\| \leq C\|s\| \) (i.e. the image of the unit ball is bounded). This is equivalent to continuous.

Let \( C^k(E) \) be set of sections all of whose local \( k \)-fold partial derivatives are continuous. Define the uniform \( C^k \)-norm

\[
\|s\|_{C^k} = \sqrt{\sup_X \sum_{j=0}^{k} \|\nabla^j s\|^2}
\]

Lemma 8 (Sobolev Embedding Lemma). For every \( l > (n/2) + k \), there is a constant \( K \) so that

\[
\|s\|_{C^k} \leq K\|s\|_l
\]

Hence the inclusion \( C^\infty(E) \subset C^l(E) \) extends to a continuous embedding

\[
L_l(E) \subset C^k(E)
\]

Thus \( C^\infty(E) = \cap_l L_l(E) \).
4 Fredholm and compact operators

Let $V$ and $W$ be Hilbert spaces. Assume all maps below are bounded linear operators.

**Definition 9.**

- $F : V \to W$ is *Fredholm* if the kernel and cokernel are finite-dimensional.
- $C : V \to W$ is *compact* if the closure of the image of the unit ball is compact.

A compact operator is a limit of finite rank operators. Intuitively a Fredholm operator is almost an isomorphism and a compact operator is almost trivial. This is made precise in the following theorem.

**Theorem 10.** $T : V \to W$ is Fredholm iff it is invertible modulo compact operators. I.e. $T : V \to W$ is Fredholm iff there is $S : W \to V$ so that $\text{Id}_V - ST$ and $\text{Id}_W - TS$ are compact operators.

As a special case, if $C : W \to W$ is compact then $\text{Id}_W - C$ is Fredholm.

**Lemma 11** (Rellich’s Lemma). For every $k$, the map $L_{k+1}(E) \to L_k(E)$ is a compact embedding.

5 Pseudodifferential Operators

Let $\Psi DO_m(E, F) \subset \operatorname{Hom}_C(C^\infty(E), C^\infty(F))$ be the vector space of pseudodifferential operators of order $m \in \mathbb{R}$. They have the following properties:

1. For an integer $m$, $DO_m(E, F) \subset \Psi DO_m(E, F)$.

2. *Symbol surjectivity:* Recall $\operatorname{Sym}_m(E, F) = \{ \sigma \in C^\infty \operatorname{Hom}(\pi^*E, \pi^*F) \mid \sigma(\rho \xi) = \rho^m \sigma(\xi), \forall \rho > 0 \}$. Here $\pi : T^*X \to X$. There is an exact sequence

   \[ \Psi DO_m(E, F) \overset{\imath}{\to} \operatorname{Sym}_m(E, F) \to 0 \]

   whose kernel is contained in $\Psi DO_{m-1}(E, F)$. This surjectivity is very important!
3. \( \Psi \text{DO's} \) have formal adjoints and the symbol of the adjoint is the adjoint of the symbol.

4. \textbf{Composition}: Composition gives a map \( \Psi \text{DO}_r(E, F) \times \Psi \text{DO}_s(F, G) \rightarrow \Psi \text{DO}_{r+s}(E, G) \) with \( \sigma_{P \circ Q} = \sigma_P \circ \sigma_Q \) (I leave it to the reader to interpret the meaning of \( \sigma_P \circ \sigma_Q \) – hint, work fiberwise!).

5. \textbf{Sobolev extension}: Sobolev spaces \( L^k(E) \) can be defined for all \( k \in \mathbb{R} \). Each \( P \in \Psi \text{DO}^m(E, F) \) extends to a bounded linear map \( P_k : L^k(E) \rightarrow \cap L^{k-m}_{l^{-m}}(F) \).

6. \textbf{Smoothing}: If \( S \in \Psi \text{DO}^{-1}(E, F) \), then for each \( k \) the composite \( \sigma_{P_k} : L^k(E) \rightarrow \cap L^{k-m}_{l^{-m}}(F) \) is a compact operator by Rellich’s Lemma.

\textbf{Definition 12.} A pseudodifferential operator is \textit{elliptic} if \( \sigma_P(\xi) : E_x \rightarrow F_x \) is an isomorphism for all \( \xi \in T^*_x X - 0 \).

Here are some corollaries:

\textbf{Corollary 13.} If \( P \in \Psi \text{DO}_m(E, F) \), then \( P_k C^\infty(E) \subset C^\infty(F) \).

\textit{Proof.} This uses Sololev extension and the Sobolev embedding theorem.
\[
P_k C^\infty(E) = P_k (\cap L^l(E)) \subset \cap P_k L^l(E) \subset \cap L^{l-m}(F) = C^\infty(F)
\]

\textbf{Corollary 14 (Existence of a parametrix).} If \( P \in \Psi \text{DO}_m(E, F) \) is elliptic, then there exists an elliptic \( Q \in \Psi \text{DO}^{-m}(F, E) \) so that
\[
\text{Id} – PQ, \ \text{Id} – QP \in \Psi \text{DO}^{-1}.
\]

\textit{Proof.} This uses symbol surjectivity and composition. Choose \( Q \) so that \( \sigma_Q = \sigma_P^{-1} \).

\textbf{Corollary 15 (Elliptic Regularity).} If \( P \) is elliptic and \( P_k(s_0) = t_0 \in C^\infty(F) \) for some \( k \), then \( s_0 \in C^\infty(E) \).

\textit{Proof.} This uses the existence of a parametrix, Sololev extension, and the Sobolev embedding theorem. Let \( Q \) be a parametrix for \( P \). Let \( m \) be the order of \( P \).
\[
s_0 = (\text{Id} – Q_{k-m} P_k)s_0 + Q_{k-m} t_0 \in L^{k+1}(E) + C^\infty(E) = L^{k+1}(E) \text{ “bootstrapping”}
\]
Thus \( s_0 \in \cap L^{k+l}(E) = C^\infty(E) \).
6 Proof of the Fundamental Theorem

Lemma 16. Let \( F : V \to W \) be a Fredholm operator. Then there is an orthogonal decomposition \( W = \ker F^* \oplus \mathrm{im} F \).

\[ \begin{align*}
W &= \overline{\mathrm{im} F} \oplus (\mathrm{im} F)^\perp \\
&= \mathrm{im} F \oplus \ker F^*
\end{align*} \]

\[ \square \]

Corollary 17. \( \ker F^* \cong \rightarrow \cok F \).

Theorem 18. Let \( P \in \Psi DO_m(E, F) \) be elliptic.

1. Each \( P_k \) is Fredholm
2. \( \ker P \hookrightarrow \ker P_k \) is an isomorphism.
3. \( \cok P_{k+1} \to \cok P_k \) is an isomorphism.
4. \( C^\infty F = \ker P^* \oplus \mathrm{im} P \).
5. \( \cok P \to \cok P_k \) is an isomorphism.

\[ \begin{align*}
\ker P_{k+1} &\cong \rightarrow \ker P_k \\
\cong &\quad \cong \\
\cok P_{k+1} &\to \cok P_k \\
\end{align*} \]

4.

- \( \ker P^* \) is f.d. by 1 and 2 and the fact that \( P^* \) is elliptic.
- \( C^\infty F = \ker P^* \oplus (\ker P^*)^\perp \)
- \( \mathrm{im} P \subset (\ker P^*)^\perp \)
• \((\ker P^*)^\perp \subset (\ker P_k^*)^\perp = \text{im } P_k\) by 2. and the Lemma.

• \((\ker P^*)^\perp = (\ker P^*)^\perp \cap \text{im } P_k \subset C^\infty F \cap \text{im } P_k \subset \text{im } P\) by elliptic regularity.

5.

\[
\begin{align*}
\ker P^* & \xrightarrow{\cong} \ker P_k^* \\
\cong & \downarrow \quad \cong \\
\cok P & \longrightarrow \cok P_k
\end{align*}
\]

where the vertical left arrow follows from the algebraic Hodge theorem.

\(\square\)

7  Pseudodifferential operators; definition of the local form

Here is the local form.

Let \(s \in C^\infty_c(\mathbb{R}^n, \mathbb{R})\). Define the Fourier transform

\[
\hat{s}(\xi) = \left(\frac{1}{2\pi}\right)^{n/2} \int e^{i\langle x,\xi \rangle} s(x) dx
\]

Two basic properties:

• \(\hat{D^\alpha \hat{s}} = \xi^\alpha \hat{s}\) (Fourier transform converts differentiation to multiplication). Here \(D^\alpha = i^{-|\alpha|} \partial^{\alpha_1} / \partial x_1^{\alpha_1} \cdots \partial^{\alpha_n} / \partial x_n^{\alpha_n}\).

• \(\hat{s} = s\) (Fourier inversion) after generalizing from compactly supported functions to Schwarz class

\[
\mathcal{S} = \{s \in C^\infty(\mathbb{R}^n, \mathbb{R}) \mid \sup_{x \in \mathbb{R}^n, \alpha, \beta} |x^\alpha D^\beta s(x)| < \infty\}
\]

Then \(\hat{\cdot} : \mathcal{S} \to \mathcal{S}\) is an isometry.

Let \(D = \sum A^\alpha(x) D^\alpha\) be a differential operator of order \(m\). Fourier inversion says

\[
s(x) = \frac{1}{(2\pi)^{n/2}} \int e^{i\langle x,\xi \rangle} \hat{s}(\xi) d\xi
\]
Then
\[ Ds(x) = \frac{1}{(2\pi)^{n/2}} \int e^{i(x,\xi)} p(x, \xi) \hat{s} (\xi) d\xi \]  \hspace{1cm} (1)

where
\[ p(x, \xi) = \sum A^\alpha(x) \xi^\alpha \]

be the \textit{total symbol} of \( D \). Note that \( p(x, \xi) \) is a polynomial of degree \( m \) in \( \xi \).

**Definition 19.** Fix \( m \in \mathbb{R} \). A function \( p(x, \xi) \) is a \textit{total symbol of order} \( m \) if for all \( \alpha, \beta \), there is a \( C \) so that for all \( x, \xi \in \mathbb{R}^n \),
\[ |D^\alpha_x D^\beta_\xi p(x, \xi)| \leq C (1 + |\xi|)^{m-|\beta|} \]

**Definition 20.** If \( p(x, \xi) \) is a total symbol of degree \( m \), then the corresponding \textit{pseudodifferential operator} \( P : \mathcal{S} \to \mathcal{S} \) is defined by equation (1).