The statement of the index theorem

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Atiyah-Singer Index Theorem (1968). Let \( D : C^\infty(E) \to C^\infty(F) \) be an elliptic differential operator. Then

\[
\text{Index } D = \text{t-Index } D
\]

Remark. 1. Index \( D \) is called the \textit{analytic index}.

2. Index \( D := \dim \ker D - \dim \text{cok } D \), where \( \text{cok } D = C^\infty F/D(C^\infty E) \).

3. t-Index \( D \) is called the \textit{topological index}. There are 3 points of view:

   (a) t-Index \( D := (-1)^n (\text{ch}(\sigma D) \text{Td}(X))[TX] \) “cohomology form of index theorem”

   (b) t-Index \( D = i_!(\sigma D) \in K^0(\text{pt}) \) “K-theory form of the index theorem”

   (c) t-Index \( D = \int_X K_{AS} \) where \( K_{AS} \) is an explicitly defined poly in the curvatures of \( X, E, \) and \( F \). “geometric form of the index theorem”

Glossary

- \( E, F \) are smooth complex vector bundles over a closed smooth manifold \( X \).
- \( C^\infty E \) and \( C^\infty F \): vector spaces of smooth sections.
- \( \sigma D \) is the \textit{symbol} of \( D \). This is a bundle \( \sigma D \downarrow T^*_C X \).
- \( \text{ch} \) and \( \text{Td} \) are the \textit{chern character} and the \textit{the Todd class}; they are characteristic classes

Four classical elliptic operators

DeRham operator, Index \( D = \chi(X) \)

Signature operator, defined for oriented manifolds, Index \( D = \text{sign}(X) \). Implies the Hirzebruch signature theorem

Dolbeault operator, defined for complex manifolds. Implies the Riemann-Roch Theorem

Dirac operator, defined for Spin manifolds
Local differential operators

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, let

$$D^\alpha = \frac{\partial^{\alpha}_1}{\partial x_{\alpha_1}} \cdots \frac{\partial^{\alpha}_n}{\partial x_{\alpha_n}} : C^\infty(\mathbb{R}^n, \mathbb{R}) \to C^\infty(\mathbb{R}^n, \mathbb{R})$$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of this differential operator.

A (linear) differential operator

$$D : C^\infty(\mathbb{R}^n, \mathbb{R}^N) = C^\infty(\mathbb{R}^n, \mathbb{R})^N \to C^\infty(\mathbb{R}^n, \mathbb{R}^M) = C^\infty(\mathbb{R}^n, \mathbb{R})^M$$

is a linear combination

$$D = \sum_\alpha A^\alpha(x)D^\alpha$$

where $A^\alpha(x)$ is an $M \times N$ matrix of real-valued functions of $n$-variables.

For $D$ of order $m$, for $x, \xi \in \mathbb{R}^n$, the (local) symbol is

$$\sigma_D(x, \xi) \in M_{M \times N}(\mathbb{R})$$

where $\sigma_D(x, \xi) = \sum_{|\alpha|=m} A^\alpha(x)\xi^\alpha$ with $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$

Definition. $D$ is elliptic if $\forall x, \forall \xi \neq 0, \sigma_D(x, \xi)$ is an isomorphism. (Hence $M = N$).

Note: For $f(x) = u(x) + iv(x) \in C^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\frac{\partial f}{\partial x_i} = \frac{\partial u}{\partial x_i} + i\frac{\partial v}{\partial x_i}$$

Example.

$$\text{Del } \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$$

- Grad $\nabla : C^\infty(\mathbb{R}^3; \mathbb{R}) \to C^\infty(\mathbb{R}^3; \mathbb{R})^3$
- Div $\nabla \cdot : C^\infty(\mathbb{R}^3; \mathbb{R})^3 \to C^\infty(\mathbb{R}^3; \mathbb{R})$
- Curl $\nabla \times : C^\infty(\mathbb{R}^3; \mathbb{R})^3 \to C^\infty(\mathbb{R}^3; \mathbb{R})$
• Laplacian \( \Delta = \nabla \cdot \nabla : C^\infty(\mathbb{R}^3; \mathbb{R}) \to C^\infty(\mathbb{R}^3; \mathbb{R}) \)

• Exterior derivative
  \[
d : C^\infty(\Lambda^1(T^*(\mathbb{R}^3))) \to C^\infty(\Lambda^2(T^*(\mathbb{R}^3)))
  \]
  \[
  1\text{-forms} \mapsto 2\text{-forms}
  \]
  \[
  \omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \mapsto d\omega = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3
  \]
  \[
  \text{where } df_i = (\partial f_i/\partial x_1)dx_1 + (\partial f_i/\partial x_2)dx_2 + (\partial f_i/\partial x_3)dx_3
  \]

Question: Does there exist an order 1 elliptic operator with \( X = \mathbb{R}^3 \)?

**Topological Manifolds**

**Definition.** A topological manifold \( X \) of dim \( n \) is a Hausdorff, 2nd countable space so that \( \forall x \in X \), \( \exists \) nbhd \( U \) of \( x \) homeo to \( \mathbb{R}^n \)

**Remark.** Any subset of \( \mathbb{R}^N \) is Hausdorff, 2nd countable.

**Theorem.** Any \( n \)-manifold is homeo to a subset of \( \mathbb{R}^{2n+1} \).

**Definition.** For an \( n \)-dim’l manifold \( X \), a chart \( h : U \to V \) is a homeo with \( U \) open in \( M \) and \( V \) open in \( \mathbb{R}^n \). \( h^{-1} : V \to U \) is a local parameterization. An atlas is a collection of charts \( \mathcal{A} = \{ h_i : U_i \to V_i \} \) so that \( \{ U_i \} \) cover \( X \).

**Smooth Manifolds**

**Definition.** Two charts \( h_1 : U_1 \to V_1 \) and \( h_2 : U_2 \to V_2 \) are smoothly related if \( h_2 h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2) \) is a diffeomorphism.

**Definition.** An atlas on \( M \) is smooth if all its charts are smoothly related.

**Proposition.** Every smooth atlas \( \mathcal{A} \) is contained in a unique max’l smooth atlas \( \mathcal{D}(\mathcal{A}) \).

In fact \( \mathcal{D}(\mathcal{A}) = \) all charts smoothly related to \( \mathcal{A} \).

**Definition.** A smooth manifold is a pair \( (X, \mathcal{D}) \) consisting of a topological manifold and a maximal smooth atlas.

**Remark.** Any smooth atlas \( (X, \mathcal{A}) \) determines an unique smooth manifold.

From now on, manifold = smooth manifold, chart = smooth chart, atlas = smooth atlas.
eg. $X = \mathbb{R}^n$ and $\mathcal{A} = \{ \text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n \}$

e.g. $X = S^n$ $\mathcal{A} = \{ h_N : (S^n - N) \rightarrow \mathbb{R}^n, h_S : (S^n - S) \rightarrow \mathbb{R}^n \}$

Smooth atlas determines manifold:
given smooth atlas $\mathcal{A} = \{ h_i : U_i \rightarrow V_i \}_{i \in \Lambda}$ on $X$,

$$X \cong \bigsqcup_{i \in \Lambda} V_i / \sim$$

where $x \in V_i \sim y \in V_j \Leftrightarrow h_i^{-1}(x) = h_j^{-1}(y)$

**Definition.** A $k$-dim'l submanifold is a subset $Y \subset X^n$ so that for every $y \in Y$, there is chart $h : U \rightarrow V$ with $y \in U$ so that $h(U \cap Y) = \mathbb{R}^k \times 0$.

**Definition.** A function $f : X \rightarrow Y$ is smooth if it is “locally smooth” if $\forall x \in X$, $\exists$ charts $(h : U \rightarrow V), (k : W \rightarrow Z)$ with $x \in U, f(x) \in W$ s.t. $kfh^{-1}$ is smooth.

**Tangent bundles and derivatives**

derivatives need tangent bundles

$X^n = \text{manifold}$

Tangent bundle $\pi : TX \rightarrow X$ where $T_xX = \pi^{-1}x$ is a $n$-dim’s v.s.

Tangent bundle of a submanifold $X$ of of $\mathbb{R}^N$

$$TX = \{(x, v) \in X \times \mathbb{R}^N | \exists \gamma : [-1, 1] \rightarrow X, \gamma(0) = x, \gamma'(0) = v \}$$

Abstract tangent bundle
If $X$ has atlas $\{h_i : U_i \rightarrow V_i\}$, then

$$TX = \bigsqcup_{i \in \Lambda} V_i \times \mathbb{R}^n \sim$$

where $(x, u)_i \sim (y, v)_j$ if $h_i^{-1}(x) = h_j^{-1}(y)$ and $d(h_jh_i^{-1})_xu = v$.

If $f : X \rightarrow Y$ is smooth, the derivative $df : TX \rightarrow TY$

$$TX \xrightarrow{df} TY \downarrow \downarrow
\text{hence } df_x : T_xX \rightarrow T_{f(x)}Y.$$
Transversality

Transversality produces manifolds

Why is sphere a manifold?

**Definition.** smooth \( f : X \to Y \) is transverse to submanifold \( L \subset Y \), if \( \forall x \in f^{-1}L \)

\[
df_x T_x X + T_{f(x)} L = T_{f(x)} Y
\]

Write \( f \pitchfork L \)

Implicit function theorem implies

**Theorem (Transversality theorem 1).** \( f \pitchfork L \iff f^{-1}L \subset X \) is a submanifold and

\[
\nu(f^{-1}L \hookrightarrow X) \cong \nu(L \hookrightarrow Y)
\]

Transversality is generic

**Theorem (Transversality theorem 2).** Let \( f : X \to Y \) be cont. and \( L \subset Y \) be a submanifold. Then \( f \simeq g \) with \( g \pitchfork L \).

Smooth vector bundles and elliptic operators

Two definitions of rank \( n \) vector bundle

**Definition (Milnor-Stasheff).** \( (p : E \to X, \forall x \in X \) v.s. structure on \( p^{-1}x \)\), so that

- \( \forall x \in X, \exists \text{nbhd } U \text{ and homeo } \phi \)

\[
p^{-1}U \xrightarrow{\phi} U \times \mathbb{R}^n
\]

so that \( p^{-1}x \to x \times \mathbb{R}^n \) is a v.s. iso.

**Definition (Steenrod).** \( (p : E \to X, \text{homeos } \mathcal{B} = \{ \phi_i : p^{-1}U_i \to U_i \times \mathbb{R}^n \}) \) so that
1. \[ \begin{array}{c}
p^{-1}U_i \\
\phi_i \\
U_i \\
\end{array} \xrightarrow{\phi_i} U_i \times \mathbb{R}^n \]

commutes

2. \( \{U_i\} \) open cover of \( X \)

3. \( \forall i, j, \exists \text{ cont } \theta_{ij} : U_i \cap U_j \rightarrow GL_n(\mathbb{R}^n) \)
   \[ \phi_i^{-1}(x, v) = \phi_j^{-1}(x, (\theta_{ij}(x)v) \]

4. \( B \) is max’l with respect to the above three properties.

**Definition** (Milnor-Stasheff). *smooth vector bundle*: require \( E \) and \( X \) are manifolds, and \( \phi \)s are diffeos.

**Definition** (Steenrod). *smooth vector bundle*: require \( X \) is manifold and \( \theta_{ij} \) are smooth.

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**Definition.** *smooth section* is smooth map \( s : X \rightarrow E \) s.t. \( p \circ s = \text{Id}_X \). \( C^\infty(E) \) is the v.s. of smooth sections.

\[ \text{e.g. } E = X \times \mathbb{R}^M, C^\infty(E) = C^\infty(X, \mathbb{R})^M \]

**Exercise.** Make precise sense of the following remark.

**Remark.** A smooth section of an \( M \)-plane bundle over an \( n \)-manifold is locally an element of \( C^\infty(\mathbb{R}^n, \mathbb{R}^M) \) with \( V \) open in \( \mathbb{R}^n \). (Explicitly use charts \( h \) and \( \phi \))

\[ C^\infty(E) \text{ is a module over } C^\infty(X, \mathbb{R}) \quad (f, s) \mapsto fs = (x \mapsto f(x)s(x)). \]
Definition of differential operators on manifolds

Let $E \to X$ and $F \to X$ be smooth vector bundle.

**Definition.** Let $Op(E, F) = \text{Hom}(C^\infty E, C^\infty F)$ (all linear maps). Define

$$Op(E, F) \times C^\infty(X, \mathbb{R}) \to Op(E, F)$$

$$(D, f) \mapsto [D, f] = Df - fD$$

Thus $[D, f]s = D(fs) - fD(s)$.

Inductively define

$$DO_m(E, F) = \subset Op(E, F)$$

$$DO_0(E, F) = \{D \in Op(E, F) \mid [D, f] = 0\}$$

$$DO_m(E, F) = \{D \in Op(E, F) \mid [D, f] \in DO_{m-1}(E, F)\}$$

$$DO(E, F) = \bigcup_m DO_m(E, F)$$

A **differential operator** is an element of $DO_m(E, F)$ for some $m$. The minimal $m$ is called the **order of $D$**.

Thus $DO_m(E, F)$ is the differential operators of order less than or equal to $m$.

**Definition.** For $s \in C^\infty E$, support $s = s^{-1}(E - 0)$

**Definition.** $D \in OP(E, F)$ is **local** if $\forall s$, support $Ds \subset$ support $s$.

**Lemma.** A **differential operator is local**.

**Proof.** By induction on $m$. True for $m = -1$. Assume true for $m - 1$. Let $D \in DO_m(E, F)$ and let $s$ be a section. $\forall$ open $U \supset$ support $s$, $\exists$ smooth $f : X^n \to [0, 1]$ with $f \equiv 0$ outside $U$ and $f \equiv 1$ in support $s$. Then $fs = s$.

$$D(s) = D(fs) = [D, f]s + fD(s)$$

So support $Ds \subset$ support $s \cup$ support $f \subset U$. Choose $U_i$ so that $\cap \overline{U}_i = $ support $s$.

**Corollary.** If $U$ is a neighborhood of $x \in X$, and $s_1$ and $s_2$ are sections that agree on $U$, then $D(s_1)x = D(s_2)x$. 

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Consider the support of $s_1 - s_2$.

**Corollary.** For an open set $U \subset X$, then is a map $DO_m(E, F) \to DO_m(E|_U, F|_U)$

Apply cutoff functions which are 1 in a neighborhood of $x \in U$ and 0 outside of $U$.

In fact, the map $U \mapsto DO_m(E|_U, F|_U)$ is a sheaf and

**Theorem** (Peetre, 1959). $D$ is local if and only if $D$ is a differential operator.

Local analysis:
- Serre-Swan theorem
- symbol as map
- clutching
- metric
- symbol

**The four classic elliptic operators**

**Complex of elliptic operators**
- DeRham operator
- Dolbeault operator
- Signature operator
- Dirac operator

**Algebraic top. of vector bundles**

1. Thom isomorphism
2. Euler Class and euler characteristic
3. splitting principle and characteristic classes
4. topological K-theory
The Index Theorem applied to the four classic elliptic operators

Analysis

Key question - why are the kernel and cokernel of an elliptic operator finite dimensional. Hodge theory . . .

Other topics

Here are possibilities:

1. The proof of the $K$-theoretic version (involves pseudo-differential operators)

2. Derivation of cohomological version from $K$-theoretic version

3. The heat kernel proof

4. Chern-Weil theory

5. APS theory

6. $L^2$-index theorem

7. applications to positive scalar curvature

8. $G$-signature theorem

9. Fixed point theorems