Anisotropic Hydrodynamics, Chiral Magnetic Effect and Holography

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Based on work with
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Chiral magnetic effect:

1. In presence of a magnetic field $B$, momenta of the quarks align along $B$

2. Topological charge induces chirality

3. Positively/negatively charged quarks move up/down (charge separation!)

4. An electric current is induced along the magnetic field $B$

$$\mathbf{J} = C \mu_5 \mathbf{B}$$

CME is a candidate for explaining an observed charge asymmetry in HIC

Does the CME hold at strong coupling?

$$\Rightarrow \text{AdS/CFT}$$

**CME in HICs:**


**Earlier papers:**

Possible dependence of the charge asymmetry on $v_2$

Event-by-event anisotropy ($v_2^{\text{obs}}$) dependence (low $p_T$)

Investigate the charge asymmetry as a function of the anisotropy $v_2^{\text{obs}}$ of the measured particles in mid-central 20–40% centrality collisions ($B \approx \text{const.}$). Consider (rare) events with different $v_2^{\text{obs}}$.

Observations:
- same-sign particles are emitted more likely in UD direction the larger $v_2^{\text{obs}}$
- same-sign particles are emitted less likely in LR direction the larger $v_2^{\text{obs}}$
  (the dependence is significantly weaker for opposite-sign particles)
- $\Rightarrow$ strong $v_2^{\text{obs}}$ dependence of the difference between UD and LR
$\Rightarrow$ charge separation depends approx. linearly on $v_2^{\text{obs}}$ (apparently in contradiction with the CME)
Overview

Goals:

i) Construct model for the chiral magnetic effect (CME) at strong coupling (AdS/CFT)
ii) In view of a possible $v_2$-dependence of the charge separation, I also study the CME in an anisotropic model

Outline:

I. CME in hydrodynamics
II. Fluid/gravity model of the CME
III. CME in anisotropic fluids
   ($v_2$-dependence)

Conclusions
Part I: CME in hydrodynamics
Hydrodynamics vs. fluid/gravity model

**Hydrodynamics**

- Multiple-charge model
  - $U(1)^n$ plasma with triangle anomalies
  - Son & Surowka (2009)
- Two-charge model
  - $U(1)_V \times U(1)_A$ plasma
  - recover CME (and other effects)

**Fluid/gravity model**

- Holographic $n$-charge model
- 5D AdS black hole geometry with $n$ U(1) charges
- Holographic two-charge model
- $n$-charge model reduced to two charges
- recover holographic CME, etc.
Hydrodynamical model with \( n \) anomalous U(1) charges:

\[ U(1)^n \text{ plasma with triangle anomalies:} \]

\[
\partial_\mu T^{\mu\nu} = F^{a\mu\nu\lambda} j^a_\lambda \quad (a = 1, 2, \ldots, n) \\
\partial_\mu j^{a\mu} = C^{abc} E^b \cdot B^c
\]

stress-energy tensor \( T^{\mu\nu} \) and U(1) currents \( j^{a\mu} \):

\[
T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + Pg^{\mu\nu} + \ldots \\
\xi^a_\omega u^\mu + \xi^a_{B\omega} \omega^\mu + \xi^a_{B\beta} B^b_{\mu} + \ldots
\]

“New” transport coefficients (not listed in Landau-Lifshitz)

- vortical conductivities \( \xi^a_\omega \) \quad Erdmenger, Haack, Kaminski, Yarom (2008)
- magnetic conductivities \( \xi^a_{B\beta} \) \quad Son & Surowka (2009)

first found in a holographic context (AdS/CFT)
First-order transport coefficients

U(1) currents:

\[ j^{a\mu} = \rho^a u^\mu + \xi^a_\omega \omega^\mu + \xi^{ab}_B B^{b\mu} + \ldots \]

vortical and magn. conductivities:

\[ \xi^a_\omega = C^{abc} \mu^b \mu^c - \frac{2}{3} \rho^a C^{bcd} \frac{\mu^b \mu^c \mu^d}{\epsilon + P} + \mathcal{O}(T^2) \]

\[ \xi^{ab}_B = C^{abc} \mu^c - \frac{1}{2} \rho^a C^{bcd} \frac{\mu^c \mu^d}{\epsilon + P} + \mathcal{O}(T^2) \]

chemical potentials

Conductivities are non-zero only in fluids with triangle anomalies!

Son & Surowka (2009), Neiman & Oz (2010)

Coffee with sugar (chiral molecules)
Two charge case \((n=2)\): \(U(1)_A \times U(1)_V\)

\(U(1)_A\) : provides chemical potential \(\mu_5\) (chirality)
\(U(1)_V\) : measures the electric current

Hydrodynamical equations:

\[
\begin{align*}
\partial_\mu T^{\mu\nu} &= F^{\nu\lambda}_V j^\lambda_
\partial_\mu j^\mu_5 &= -\frac{C}{4} F^{\nu\lambda}_\mu F^{\lambda\nu}_V j^\mu_5
\partial_\mu j^\mu &= 0
\end{align*}
\]

axial gauge field switched off! \((A^A_\mu = 0)\)

Constitutive equations:

\[
\begin{align*}
j^\mu &= \rho u^\mu + \kappa_\omega \omega^\mu + \kappa_B B^\mu
j^\mu_5 &= \rho_5 u^\mu + \xi_\omega \omega^\mu + \xi_B B^\mu
\end{align*}
\]

\(CME\) coefficient

Identifications:

\[
\begin{align*}
A^A_\mu &= A^1_\mu, & A^V_\mu &= A^2_\mu
j^\mu_5 &= j^{1\mu}, & j^\mu &= j^{2\mu}
\mu_5 &= \mu^1, & \mu &= \mu^2,
\rho_5 &= \rho^1, & \rho &= \rho^2,
\kappa_\omega &= \xi_\omega^2, & \kappa_B &= \xi_B^{22},
\xi_\omega &= \xi_1^\omega, & \xi_B &= \xi_1^B
\end{align*}
\]

\(C\)-parity allows for:

\[
\begin{align*}
C^{111} &= C/3 \quad (AAA)
C^{122} &= C^{221} = C^{212} = C/3 \quad (AVV)
C^{121} &= C^{211} = C^{112} = 0 \quad (VAA)
C^{222} &= 0 \quad (VVV)
\end{align*}
\]
(Chiral) magnetic and vortical effects

constitutive equations:

\[ j^\mu = \rho u^\mu + \kappa_\omega \omega^\mu + \kappa_B B^\mu \]
\[ j_5^\mu = \rho_5 u^\mu + \xi_\omega \omega^\mu + \xi_B B^\mu \]

transport coefficients (conductivities):

<table>
<thead>
<tr>
<th>CVE</th>
<th>QVE</th>
<th>CME</th>
<th>CSE (QME)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_\omega / \mu = 2C \mu_5 \left(1 - \frac{\mu \rho}{\epsilon + P} \left[1 + \frac{\mu_5^2}{3\mu^2}\right]\right) )</td>
<td>( \xi_\omega / \mu = C \mu \left(1 - 2\frac{\mu_5 \rho_5}{\epsilon + P} \left[1 + \frac{\mu_5^2}{3\mu^2}\right]\right) )</td>
<td>( \kappa_B = C \mu_5 \left(1 - \frac{\mu \rho}{\epsilon + P}\right) )</td>
<td>( \xi_B = C \mu \left(1 - \frac{\mu_5 \rho_5}{\epsilon + P}\right) )</td>
</tr>
</tbody>
</table>

\( C = \) chiral since CVE, CME prop. to chiral chemical potential \( \mu_5 \)

\( Q = \) quark since QVE, QME prop. to quark chemical potential \( \mu \)

(QME also called chiral separation effect (CSE))

\( \mu \tilde{\omega} \) creates an effective magnetic field Kharzeev and Son (2010)
Part II: Fluid/gravity model of the CME

Kalaydzhyan & I.K., PRL 106 (2011) 211601
Hydrodynamics vs. fluid/gravity model

Hydrodynamics

Multiple-charge model

$U(1)^n$ plasma with triangle anomalies
Son & Surowka (2009)

Two-charge model

$U(1)_V \times U(1)_A$ plasma
recover CME (and other effects)

Fluid/gravity model

Holographic $n$-charge model

5D AdS black hole geometry with $n$ U(1) charges

Holographic two-charge model
$n$-charge model reduced to two charges
recover holographic CME, etc.
Gravity: Holographic computation

Strategy: quark-gluon plasma is strongly-coupled → use AdS/CFT to compute the transport coefficients relevant for the anomalous effects (CME, etc.)

- find a 5d charged AdS black hole solution with several U(1) charges

- duality:

\[(m, q_a) \leftrightarrow (T, \mu_a)\]

mass \(m\)  
U(1) charges \(q_a\)  
Hawking temperature \(T \sim r_+\)

\[\mu^a \equiv A^a_0(r_+) - A^a_0(\infty)\]

- use fluid-gravity methods to holographically compute the transport coefficients \(\kappa_\omega, \kappa_B, \xi_\omega, \xi_B\) (i.e. CME and other effects)
AdS black hole solution with multiple U(1) charges

Five-dimensional $U(1)^n$ Einstein-Maxwell theory with cosmological term:

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R + 12 - F_{MN}^a F^{aMN} + \frac{S_{abc}}{6\sqrt{-g}} \varepsilon^{PKLMN} A^a_P F^K_l F^c_{MN} \right]$$

Fields:
- metric $g_{MN}$ ($M, N = 0, \ldots, 4$)
- $n$ U(1) gauge fields $A^a_M$ ($a = 1, \ldots, n$)
- cosmological constant $\Lambda = -6$

The information of the anomalies is encoded in the Chern-Simons coefficients

$$S_{abc} = 4\pi G_5 C_{abc}$$

Son & Surowka (2009)
Boosted AdS black hole solution

5d AdS black hole solution ($0^{th}$ order solution):

$$ds^2 = -f(r)u_\mu u_\nu dx^\mu dx^\nu - 2u_\mu dx^\mu dr + r^2 (\eta_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu$$

$$A^a = (A_0^a(r) u_\mu + A_\mu^a) dx^\mu$$

with

$$f(r) = r^2 - \frac{m}{r^2} + \sum_a \frac{(q_a)^2}{r^4}$$

$$A_0^a(r) = -\frac{\sqrt{3} q^a}{2 r^2}$$

$n$ charges $q^a$

$u_\mu$ = four-velocity of the fluid

background gauge field added (needed to model external B-field)!
First-order transport coefficients

We use the standard procedure of Bhattacharyya et al. (2008) to holographically compute the transport coefficients $\xi^a_{\omega}$ (Torabian & Yee (2009) for $n=3$) and $\xi^{ab}_{B}$:

1. Vary 4-velocity and background fields (up to first order):

$$u_\mu = (-1, x^\nu \partial_\nu u_i), \quad A_\mu^a = (0, x^\nu \partial_\nu A_i^a)$$

The boosted black-brane solution ($0^{th}$ order sol.) is no longer an exact solution, but receives higher-order corrections.

Ansatz for first-order corrections:

$$ds^2 = (-f(r) + \tilde{g}_{tt}) \, dt^2 + 2(1 + \tilde{g}_{tr}) \, dtdr + r^2(dx^i)^2 + \tilde{g}_{ij} dx^i dx^j - 2x^\nu \partial_\nu u_i dr dx^i + 2\left((f(r) - r^2) x^\nu \partial_\nu u_i + \tilde{g}_{ti}\right) \, dt dx^i,$$

$$A^a = (-A_0^a(r) + \tilde{A}_t^a) \, dt + \left(A_0^a(r)x^\nu \partial_\nu u_i + x^\nu \partial_\nu A_i^a + \tilde{A}_i^a\right) dx^i$$

Need to determine first order corrections $\tilde{g}_{tt}, \tilde{g}_{tr}, etc.$
2. Solve equations of motion (system of Einstein-Maxwell equations) and find the first-order corrections to the metric and gauge fields:

\[ \tilde{g}_{tr} = \tilde{g}_{tt} = \tilde{A}_i^a = 0 \]

\[ \tilde{g}_{ii}(r) = f(r) \int_0^r \frac{1}{r' (f(r'))^2} \left( \int_{r_+}^{r'} dr'' I(r'') - r_+ f'(r_+) C_i \right) \]

\[ \tilde{A}_i^a(r) = \int_0^r \frac{1}{r' f(r')} \left[ Q_i^a(r') - Q_i^a(r_H) - C_i r + A_0^a(r_+) + r' \tilde{g}_{ii}(r') A_0^{a'}(r') \right] \]

(lengthy calculation, \( I(r) \), \( Q_i^a(r) \), \( C_i \) functions of \( A_0^a(r) \), \( f(r) \), \( u_i \))

3. Read off energy-momentum tensor and \( U(1) \) currents from the near-boundary expansion of the first-order corrected background (e.g. Fefferman-Graham coordinates):

\[ T_{\mu\nu} = \frac{g_{\mu\nu}^{(4)}(x)}{4\pi G_5} + c.t \]

\[ j_\alpha^\mu = \frac{\eta^{\mu\nu} A_{\alpha\nu}^{(2)}(x)}{8\pi G_5} + \tilde{j}_\alpha^\mu \]

\[ ds^2 = \frac{1}{z^2} \left( g_{\mu\nu}(z, x) dx^\mu dx^\nu + dz^2 \right), \]

\[ g_{\mu\nu}(z, x) = \eta_{\mu\nu} + g_{\mu\nu}^{(2)}(x) z^2 + g_{\mu\nu}^{(4)}(x) z^4 + ... \]

\[ A_\mu^a(z, x) = A_\mu^a(0)(x) + A_\mu^a(2)(x) z^2 + ... \]
First-order transport coefficients (cont.)

4. Determine the vortical and magnetic conductivities $\xi^a_\omega$ and $\xi^{ab}_B$

\[
T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \ldots
\]

\[
j^{a\mu} = \rho^a u^\mu + \xi^a_\omega \omega^\mu + \xi^{ab}_B B^{b\mu} + \ldots
\]

use identities (from zeroth order solution):

\[
P \equiv m/16\pi G_5
\]

\[
\rho_a \equiv \sqrt{3} q_a / 16\pi G_5 \quad \Rightarrow \quad \frac{\sqrt{3} q^a}{4m} = \frac{\rho^a}{\epsilon + P} \quad (\epsilon = 3P)
\]

transport coefficients:

\[
\xi^a_\omega = \frac{4}{16\pi G_5} \left( S^{abc} \mu^b \mu^c - \frac{2}{3} \frac{\rho^a}{\epsilon + P} S^{bcd} \mu^b \mu^c \mu^d \right)
\]

\[
\xi^{ab}_B = \frac{4}{16\pi G_5} \left( S^{abc} \mu^c - \frac{1}{2} \frac{\rho^a}{\epsilon + P} S^{bcd} \mu^c \mu^d \right)
\]

\[
\mu^a \equiv A^a_0(r_H) - A^a_0(\infty)
\]

\[
S_{abc} = 4\pi G_5 C_{abc} \quad \Rightarrow \quad \text{We recover the hydrodynamic result!}
\]
Holographic magnetic and vortical effects

Using the same identifications as in hydrodynamics, but now for the holographically computed transport coefficients, we get

\[
\kappa_{\omega} = 2 C \mu \mu_5 \left( 1 - \mu \frac{\sqrt{3} q}{4m} \left[ 1 + \frac{\mu_5^2}{3 \mu^2} \right] \right), \quad \kappa_B = C \mu_5 \left( 1 - \mu \frac{\sqrt{3} q}{4m} \right)
\]

\[
\xi_{\omega} = C \mu^2 \left( 1 - 2 \mu_5 \frac{\sqrt{3} q_5}{4m} \left[ 1 + \frac{\mu_5^2}{3 \mu^2} \right] \right), \quad \xi_B = C \mu \left( 1 - \mu_5 \frac{\sqrt{3} q_5}{4m} \right)
\]

Result: CME, CVE, etc. are realized in an n-charged AdS black hole model (plus background gauge field), when appropriately reduced to a two-charge model (n=2).

Other AdS/QCD models:
- Lifschytz and Lippert (2009), Yee (2009),
Part III: CME in anisotropic fluids

Possible dependence of the charge asymmetry on $v_2$

Event-by-event anisotropy ($v_2^{obs}$) dependence (low $p_T$)

Investigate the charge asymmetry as a function of the anisotropy $v_2^{obs}$ of the measured particles in mid-central 20–40% centrality collisions ($B \approx const.$). Consider (rare) events with different $v_2^{obs}$.

Observations:
- same-sign particles are emitted more likely in UD direction the larger $v_2^{obs}$
- same-sign particles are emitted less likely in LR direction the larger $v_2^{obs}$ (the dependence is significantly weaker for opposite-sign particles)
- => strong $v_2^{obs}$ dependence of the difference between UD and LR
- => charge separation depends approx. linearly on $v_2^{obs}$ (apparently in contradiction with the CME)
Build-up of the elliptic flow and momentum anisotropy

Central question: In anisotropic fluids, does the chiral conductivity depend on \( v_2 \)?

Sketch of the time-evolution of the momentum anisotropy \( \varepsilon_P \):

\[
\varepsilon_P = \frac{P_T - P_L}{P_T + P_L}
\]

at freeze-out:
\[ v_2 \approx \varepsilon_P / 2 \]

Our model describes a state after thermalization with unequal pressures \( P_T \neq P_L \).
We do not model the full evolution of \( \varepsilon_P \).
Hydrodynamics of an anisotropic fluid

Anisotropic fluid with \( n \) anomalous U(1) charges

stress-energy tensor \( T^{\mu\nu} \) and U(1) currents \( j^{\alpha\mu} \):

\[
T^{\mu\nu} = (\epsilon + P_T)u^\mu u^\nu + P_T g^{\mu\nu} - (P_T - P_L)v^\mu v^\nu + \tau^{\mu\nu}
\]

\[
j^{\alpha\mu} = \rho^\alpha u^\mu + \nu^{\alpha\mu}
\]

orthogonality and normalization:

\[
u_{\mu} u^\mu = -1, \quad v_{\mu} v^\mu = 1, \quad u_{\mu} v^\mu = 0
\]

local rest frame:

\[
u^\mu = (1, 0, 0, 0) \quad v^\mu = (0, 0, 0, 1)
\]

\[
T^{\mu\nu} = \text{diag}(\epsilon, P_T, P_T, P_L)
\]

thermodynamic identity:

\[
\epsilon + P_T = Ts + \mu \rho
\]
Chiral magnetic and vortical effects

Anisotropic fluid with one axial and one vector U(1)

Repeating the hydrodynamic computation of Son & Surowka (for \( n=2 \)), we find the following result for the chiral magnetic effect:

\[
\Delta j^\mu = \kappa_B B^\mu, \quad \kappa_B = C \mu_5 \left( 1 - \frac{\mu \rho}{\epsilon + P_T} \right)
\]

or, for small \( \varepsilon_p \),

\[
\kappa_B \approx C \mu_5 \left( 1 - \frac{\mu \rho}{\epsilon + \bar{P}} \left[ 1 - \frac{\varepsilon_p}{6} \right] \right)
\]

as before \( \bar{P} = \frac{2P_T + P_L}{3} \)

multiple charge case (\( n \) arb.):

\[
\xi_B^{ab} = C^{abc} \mu^c - \frac{1}{2} \frac{\rho^a}{\epsilon + P_T} C^{bcd} \mu^c \mu^d
\]

simplifying assumptions:

\[
\partial_\mu v^\mu = 0, \quad \mu_\mu \partial_\mu \Delta = 0 \\
(\Delta = P_T - P_L)
\]
Boosted anisotropic AdS black hole solution (w/ \( n \) U(1)’s)

5d AdS black hole solution (ansatz):

\[
ds^2 = \left( r^2 w_T(r) P_{\mu\nu} - f(r) u_\mu u_\nu \right) dx^\mu dx^\nu - 2 u_\mu dx^\mu dr - r^2 (w_T(r) - w_L(r)) v_\mu v_\nu dx^\mu dx^\nu,
\]

\[
A^a = (A_0^a(r) u_\mu + A_\mu^a) dx^\mu
\]

asymptotic solution (close to the boundary):

\[
f(r) = r^2 - \frac{m}{r^2} + \sum_a \frac{(q_a)^2}{r^4} + \mathcal{O}(r^{-6})
\]

\[
w_T(r) = 1 + \frac{m\zeta}{4r^4} + \mathcal{O}(r^{-8})
\]

\[
w_L(r) = 1 - \frac{m\zeta}{2r^4} + \mathcal{O}(r^{-8})
\]

\[
A_0^a(r) = -\frac{\sqrt{3}q^a}{2r^2} + \mathcal{O}(r^{-10})
\]

\[
\varepsilon = 2P_T + P_L \quad \rho^a = \frac{\sqrt{3}q^a}{16\pi G_5}
\]

\[
P_T = \frac{m(1+\zeta)}{16\pi G_5} , \quad P_L = \frac{m(1-2\zeta)}{16\pi G_5}
\]

no analytic solution \(\rightarrow\) numerical solution
Numerical solution for the AdS black hole background

Shooting techniques provide numerical plots for the functions $f(r), A_0(r), w_T(r), w_L(r)$ for $\zeta=10$:

outer horizon: $r_+ = 1$
First-order transport coefficients

The holographic computation of the transport coefficients is very similar to that in the isotropic case.

Magnetic conductivities (result):

\[
\xi_{ab} = \frac{4}{16\pi G_5} \left( S^{abc} \mu^c - \frac{1}{2} C(r_+) S^{bcd} \mu^c \mu^d \right)
\]

\[
\mu^a \equiv A^a_0(r_H) - A^a_0(\infty), \quad S_{abc} = 4\pi G_5 C_{abc}
\]

\[
C(r_+) = \frac{r_+ \partial_{r_+} A^a_0(r_+)}{r_+ (f'(r_+) - 4 \sum_a A^a_0(r_+) \partial_{r_+} A^a_0(r_+) - \frac{1}{w_L(r_+)^{1/2}}}
\]

\[
\frac{1}{\sqrt{3}} \frac{1}{4m} \frac{1}{1 + \frac{1}{4}\zeta} q_a = \frac{\rho^a}{\varepsilon + P_T}
\]

⇒ find agreement with hydrodynamics if the orange factors agree

(needs to be shown numerically)
Numerical agreement with hydrodynamics

Numerical plot $w_L(r_\perp)$ as a function of the anisotropy:

agreement with hydrodynamics since

$$w_L(r_\perp) = \left(1 + \frac{1}{4}\zeta\right)^2$$
Conclusions

I presented two descriptions of the CME (and related effects) in

a) isotropic plasmas ($P = P_T = P_L$):

i) hydrodynamic model: $U(1)_A \times U(1)_V$ fluid with triangle anomaly

ii) holographic fluid-gravity model: 5d AdS-Reissner-Nordstrom-like solution with two U(1) charges

Agreement was found between both models.

b) anisotropic plasmas ($P_T \neq P_L$):

- experimental data suggests possible $v_2$ – dependence of the charge separation
- Does the chiral magn. conductivity $\kappa_B$ depend on $v_2$?
- we constructed anisotropic versions of the above $U(1)_A \times U(1)_V$ models and found

$$\kappa_B \approx C \mu_5 \left( 1 - \frac{\mu \rho}{\varepsilon + P} \left[ 1 - \frac{\varepsilon_P}{6} \right] \right)$$

where

- $C$ is a constant,
- $\mu$ is the electric charge density,
- $\rho$ is the energy density,
- $\varepsilon$ is the energy density,
- $P$ is the pressure,
- $\kappa_B$ is the conductivity,
- $\varepsilon_P$ is a parameter related to the hadronic matter.

$$\tilde{P} = \frac{2P_T + P_L}{3}$$

Is the observed charge asymmetry a combined effect of the CME (1st term in $\kappa_B$) and the dynamics of the system (2nd term)?