Chiral anomaly and magnetic fields in the Standard Model plasma

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P- and CP-Odd Effects in Hot and Dense Matter
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In this talk I will demonstrate that

- At high temperatures / densities the equilibrium state of the **Standard Model** plasma is not homogeneous and isotropic but should contain large scale magnetic fields

- The description of evolution of helical magnetic fields at high temperatures (by any mechanism) requires additional "degree of freedom"

- Both results are based *solely on the Standard Model physics*

Based on the recent works:

- A. Boyarsky, J. Fröhlich, **O.R.**
Can homogeneous and isotropic plasma spontaneously break translational invariance?

[1204.3604]
Properties of the equilibrium system are characterized by its temperature and the values of conserved charges.

In the Standard Model at $T < 100$ GeV (when electroweak symmetry is broken) there are 4 conserved charges:

- Baryon number $B$
- Three flavour lepton numbers $L_e, L_\mu, L_\tau$

Additionally the plasma is electrically neutral.

Plasma breaks Lorentz invariance down to 3-dimensional symmetry.
Static magnetic fields in plasma

- Effective action of the static electromagnetic fields has the form

$$\mathcal{F}[A] = \frac{1}{2} \int d^3 p \ A_i(p^\dagger) \Pi_{ij}(p) A_j(-p^\dagger) + \mathcal{O}(A^3)$$  

(magnetic field $\vec{B} = \nabla \times \vec{A}$)

- **Polarization operator** $\Pi_{ij}$ should be rotation invariant and gauge invariant (i.e. transversal: $p_i \Pi_{ij} = 0$). The most general form:

$$\Pi_{ij}(p^\dagger) = (p^2 \delta_{ij} - p_i p_j) \Pi_1(p^2) + i\epsilon_{ijk} p^k \Pi_2(p^2)$$

  parity-even part  

  parity-odd part

- $\Pi_1$ is a renormalization of the electric charge, we will forget about it from now on ($\Pi_1 = 1$)
- here and below we will speak only about $\Pi_2(0)$ that we denote simply by $\Pi_2$
In coordinate space $\Pi_2 \neq 0$ this leads to a Chern-Simons term:

$$\mathcal{F}[A] = \frac{1}{2} \int d^3 x \left( B^2 - \Pi_2 A \cdot B \right)$$

The Chern-Simons term
- contains less derivatives than $(\nabla \times A)^2$
- can be both positive and negative

The matrix $\Pi_{ij}$ has a negative eigenvalue for

$$p < \Pi_2$$

Long-range magnetic fields with $p < \Pi_2$ will be generated
Maximally helical configuration

- The unstable mode will have a form

\[ \vec{A}(\vec{x}) = A_0 \left( \cos(pz), \sin(pz), 0 \right) \]

- The magnetic field

\[ \vec{B}(\vec{x}) = -p \vec{A}(\vec{x}) \]

— is maximally helical

- On this configuration \( \vec{B}^2 = p \vec{A} \cdot \vec{B} \) and are homogeneous

- The effective action:

\[ \mathcal{F}[A] = \frac{1}{2} \int d^3 x \left( p^2 - p \Pi_2 \right) A_0^2 < 0 \]

for \( p < \Pi_2 \)
Origin of Chern-Simons term

- Chern-Simons terms are usually prohibited by discrete symmetries ($P, CP, CPT$)

- The origin of this term?

- $P, CP, CPT$ are broken by non-zero chiral charges of chiral fermions (by non-zero chemical potentials $\mu_L$ and $\mu_R$)

- If number of left particles $\neq$ the number of right particles (i.e. they have different chemical potentials $\mu_R \neq \mu_L$) then

$$\Pi_2 = \frac{\alpha}{\pi} \Delta \mu$$

- Vilenkin (1978)
- Redlich & Wijewardhana (1985);
- Fröhlich et al. (1998–2001)
- Joyce & Shaposhnikov (1997)
Chern–Simons term and axial anomaly

- In plasma with the different number of left and right particles

\[ \Pi_2 = \frac{\alpha}{\pi} \Delta \mu \]

\[ A_i(\vec{p}) A_j(-\vec{p}) \]

- This diagram is related to axial anomaly

\[ \Delta \mu \gamma_0 \gamma_5 \]

\[ A_i(\vec{p}) A_j(-\vec{p}) \]

\[ \langle j_5^\mu \rangle \]
Different number of left and right chiral particles?

\[
\frac{1}{\exp\left(\frac{p-\mu_L}{T}\right)+1} \neq \frac{1}{\exp\left(\frac{p-\mu_R}{T}\right)+1}
\]
Chirality flipping processes are related to fermion’ Yukawa (or mass).

Although $T \gg m$ and these reactions are suppressed as $(m/T)^2$ as compared to chirality-preserving reactions after long time they will wash out $\Delta \mu$:

$$\frac{\Delta \mu}{dt} = -\Gamma_f \Delta \mu$$
Examples: early Universe

- Starting from $T \sim 80$ TeV chirality flipping processes are in equilibrium ($\Gamma_{\text{flip}}(T) \gg H(T) = T^2/M_*$)

\begin{align*}
\text{Symmetric phase:} \\
\bullet & \quad \Gamma_{\text{high-temp}} \sim \frac{80\,\text{TeV}}{M_*} T \\
\text{Broken phase:} \\
\bullet & \quad \Gamma_{\text{EM}} \propto \alpha^2 T \left(\frac{m_e}{3T}\right)^2 \\
\bullet & \quad \Gamma_W \propto G_F^2 T^5 \left(\frac{m_e}{3T}\right)^2
\end{align*}
Although \( \left( \frac{m_e}{80 \text{ TeV}} \right)^2 \sim 10^{-17} \) chirality flipping reactions are in thermal equilibrium for \( T < 80 \text{ TeV} \) and drive \( \mu_L - \mu_R \) to zero \textit{exponentially fast} (suppression of at least \( e^{-1000} \) over one Hubble time).

Only \textit{non-equilibrium relaxation} of initial \( \Delta \mu(t) \) is possible? This relaxation can be “slow”…

Equilibrium state is always \( \mu_L = \mu_R \)?

\textbf{No!}

It is possible to have \textit{equilibrium} difference of chemical potentials!

This \textit{does not require} super-high temperatures (can even happen at zero temperature but finite density!)
Weak corrections

- Weak corrections lead to the **change of dispersion relations** (shift of chemical potentials) of left/right particles. It is crucial that chirality flipping processes are in equilibrium.

The resulting $\mu_L - \mu_R$ is proportional to the **asymmetry** of all fermions, running in the loops.

- Asymmetry $n_\psi - n_{\bar{\psi}} \propto$ global charges $(B, L_e, L_\mu, L_\tau)$.
\[\Pi_2 = \frac{\alpha}{2\pi} G_F \times (c_1 \text{ baryon number} + c_2 \text{ lepton numbers}) \neq 0\]
Purely within the Standard Model:

- Chern-Simons term should be added to the Standard Model Lagrangian and finite densities of lepton and/or baryon number

- Homogeneous and isotropic ground state of primordial plasma is **unstable towards developing a long range magnetic fields**

- The origin of this effect is the parity-breaking character of weak interactions and chiral anomaly
From equilibrium to non-equilibrium

This result was purely equilibrium computation ↔ property of the ground state of the Standard Model plasma at finite temperature and/or density.

How does this effect exhibit itself in the non-equilibrium case? How does a deviation from equilibrium look like? Let us consider the case without non-zero global charges for simplicity.

Reminder: dynamics of the magnetic field is described by the MHD equations

$$\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times B) + \frac{1}{\sigma} \nabla^2 B$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{v}) = -\nabla p + \nu \nabla^2 \mathbf{v} + (\nabla \times B) \times B$$

(in the absence of $\Delta \mu$!)
Magnetic diffusion

- Ohmic dissipation of magnetic energy — magnetic diffusion:

\[ \frac{\partial B}{\partial t} = \frac{1}{\sigma} \nabla^2 B \]

- Generically magnetic fields at scales below magnetic diffusion scale

\[ \ell_\sigma \sim \sqrt{\frac{t}{\sigma}} \]

are erased over a characteristic time \( t \)

- Example: early Universe: — magnetic diffusion scale today \( \sim 10^{13} \) cm. Horizon of EW epoch is stretched to \( \sim 10^{15} \) cm

What changes in the presence of \( \Delta \mu \)?
Maxwell equations

The presence of a difference of chemical potential of left and right fermions leads to additional terms in the effective Lagrangian for electromagnetic fields – Chern-Simons term.

As a result, Maxwell equations contain a current, proportional to $\Delta \mu$ – MHD turns into chiral MHD:

\[ \text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \text{curl } \vec{B} = \sigma \vec{E} + \frac{\alpha}{\pi} \Delta \mu \vec{B} \]

Chiral magnetic effect

References:
- Kharzeev’11
- Vilenkin (1978)
- Redlich & Wijewardhana (1985);
- Fröhlich et al. (1998–2001)
- Joyce & Shaposhnikov (1997)
**New degree of freedom**

- **In addition**, $\Delta \mu$ should be allowed to become dynamical:
  
  because $\partial_{\mu} j_{5}^{\mu} \propto E \cdot B$

  \[
  \text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}
  \]

  \[
  \text{curl } \vec{B} = \sigma \vec{E} + \frac{2\alpha}{\pi} \int d^3 x \vec{E} \cdot \vec{B}
  \]

  \[
  \frac{\partial \Delta \mu}{\partial t} \propto \frac{2\alpha}{\pi} \int d^3 x \vec{E} \cdot \vec{B} - \Gamma_{\text{flip}} \Delta \mu
  \]

- **Naively:**
  - Without $B$ chirality flipping reactions drive $\Delta \mu \to 0$
  - Without $\Delta \mu$ finite conductivity drives $B \to 0$
Instability

■ Maxwell equations with $\Delta \mu$ are unstable:

$$\frac{\partial B}{\partial t} = \frac{1}{\sigma} \nabla^2 B + \frac{\alpha \Delta \mu}{\pi} \text{curl} B$$

magnetic diffusion

■ Exponential growth for $k < \Delta \mu$ (for one of the circular polarizations depending on the sign of $\Delta \mu$) — generation of helical magnetic fields

$$B_\pm = B_0 \exp\left( -\frac{k^2}{\sigma} t \pm \frac{\alpha k \Delta \mu}{\pi \sigma} t \right)$$

■ Exponential growth for $k < \frac{\alpha}{\pi} \Delta \mu$
Helical magnetic fields in presence of $\Delta \mu$

- If there are electromagnetic fields in plasma we have

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\sigma} \nabla^2 \vec{B} + \frac{\alpha \Delta \mu}{\pi \sigma} \nabla \times \vec{B}$$

- If $\Delta \mu \neq 0$ – magnetic field grows and

$$\frac{d(\Delta \mu)}{dt} = -(c\Delta \alpha) \frac{2}{V} \int_V d^3x \vec{E} \cdot \vec{B} - \Gamma_f \Delta \mu$$

O.R. with A. Boyarsky, J. Fröhlich PRL (2012)

- One cannot have $\Delta \mu = 0$ if $\int \vec{E} \cdot \vec{B} \neq 0$
Consider sharply peaked at \( k_0 \) maximally helical field

\[
\frac{d\Delta \mu}{dt} = -\rho_B \left( \Delta \mu - \Delta \mu_{tr} \right) - \Gamma_{\text{flip}} \Delta \mu
\]

Assume \( \rho_B \gg \Gamma_{\text{flip}} \)

\[
\frac{d\rho_B}{dt} = \frac{\rho_B}{t_\sigma} \left( \frac{\Delta \mu}{\Delta \mu_{tr}} - 1 \right)
\]

where \( \Delta \mu_{tr} = \frac{2\pi k_0}{\alpha} \)

Large \( \rho_B \) drives \( \Delta \mu \) to an attractor solution \( \Delta \mu_{tr} \).

Electric conductivity of the plasma is finite \( (t_\sigma = \frac{2\sigma}{k_0^2}) \) but magnetic diffusion is compensated by the presence of \( \Delta \mu \).
Relative change of magnetic energy of a single mode in the chiral MHD
In case of two modes the helicity gets transferred from the shorter one to the longer one.

Chemical potential follows the wave-number of the mode with higher helicity.
Evolution of chemical potential

\[
\frac{\Delta \mu}{T} \approx \frac{5 \times 10^{-5}}{10^{0.1}, 10^{0.5}, 10^{1}, 10^{1.5}, 10^{2}}
\]

Conformal time \(\lg(M_*/T)\)

Process continues while \(\rho_B \gg \Gamma_{\text{flip}}\) (recall that \(\rho_B \propto \rho_B\))
Evolution of helicity spectrum

Process continues while $\Gamma_B \gg \Gamma_{\text{flip}}$ (recall that $\Gamma_B \propto \rho_B$)
Evolution of chemical potential

Continuous initial spectrum with $\mathcal{H}_k \propto k$ and fraction of magnetic energy density $5 \times 10^{-5}$ (blue) or $5 \times 10^{-4}$ (green). Red – evolution without flip
Evolution of helicity spectrum

Inverse cascade without turbulence!

T=150 MeV

T=100 GeV

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MAGNETIC FIELDS AND CHIRAL ASYMMETRY
Purely within the Standard Model:

- Chern-Simons term should be added to the Standard Model Lagrangian and finite densities of lepton and/or baryon number

- Homogeneous and isotropic ground state of primordial plasma is unstable towards developing a long range magnetic fields
Purely within the Standard Model:

- Evolution of relativistic plasma with long-wavelength magnetic fields is not described by the standard MHD equations (as was previously believed)

- Additional IR degree of freedom (the difference of chemical potentials of electrons) should be made dynamical and significantly affects evolution.

- This new degree of freedom:
  - Leads to conservation of helicity of magnetic fields
  - Partially compensates dissipation due to magnetic diffusion. Creates “inverse cascade” in spectrum of $B$-field (without turbulence)
  - Preserves chemical potential difference while $\rho_B \gg \Gamma_{\text{flip}}$ (e.g. down to $T \sim$ few MeV for electrons)
Consequences for early Universe

- Cosmological magnetic fields of horizon size in the electroweak epoch purely within the Standard Model

- QCD phase transition (can make it first order)

- BBN

- Dark matter production (to follow)

- This effect make a survival of helical magnetic fields (generated at and prior to 100 GeV) possible, as the inverse cascade transfers energy at larger and larger scales (e.g. following horizon)

References:
- Schwarz & Stuke (2009)
- Vachaspati, Davidson, Grasso & Riotto, …
Thank you for your attention!
Massless fermions can be left and right-chiral (left and right moving):

\[(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m^0 & i(\partial_t + \vec{\sigma} \cdot \vec{\nabla}) \\ i(\partial_t - \vec{\sigma} \cdot \vec{\nabla}) & -m^0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0\]

where \(\gamma_5 \psi_{R,L} = \pm \psi_{R,L}\) and \(\gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3\)

Number of left \(N_L = \int d^3x \psi_L^\dagger \psi_L\) and right \(N_R = \int d^3x \psi_R^\dagger \psi_R\) particles is conserved independently \(N_L + N_R\) and \(N_L - N_R\) are conserved independently in the free theory.
The situation changes if gauge fields are present. Gauge interactions respect chirality \( D_\mu = \partial_\mu + eA_\mu \)...

\[
\begin{pmatrix}
0 & i(D_t + \vec{\sigma} \cdot \vec{D}) \\
(i(D_t - \vec{\sigma} \cdot \vec{D}) & 0
\end{pmatrix}
\begin{pmatrix}
\psi_L \\
\psi_R
\end{pmatrix} = 0
\]

... but the difference of left and right-movers is still not conserved:

\[
\frac{d(N_L - N_R)}{dt} = \int d^3 \vec{x} \left( \partial_\mu j_5^\mu \right) = \frac{e^2}{4\pi^2} \int d^3 \vec{x} \vec{E} \cdot \vec{B}
\]
Chiral anomaly at finite fermion densities

- Chiral anomaly for degenerate Fermi gas

\[ \delta N_{L,R}(t) = \int dt \, \dot{N}_{L,R}(t) = \mp \int dt \frac{\alpha}{\pi} \int d^3 x \, E \cdot B \]

- Fermions occupy levels up to Fermi levels \( \mu_{L,R} \).

- The energy change:

\[ \delta \mathcal{E} = \delta N_L \mu_L + \delta N_R \mu_R = \frac{\alpha}{2\pi} \Delta \mu \int d^3 x \, A \cdot B \]

- Free energy density:

\[ \delta \mathcal{F} = \frac{\alpha}{2\pi} \Delta \mu A \cdot B \]

Nielsen & Ninomiya (1983); Rubakov (1986)